A Novel Numerical Optimization technique to Control the Accuracy of Semi-Analytical Methods for Solving Volterra Integral Equations with Discontinuous Kernel

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Introduction

The linear and nonlinear VIEs with discontinuous kernel have been applied to modelize the load forecast in EPS with renewable generation. Thus finding numerical and semi-analytical methods for solving this problem and validating the numerical results to forecast the load leveling problem are among challenging problems in applied mathematics.

The Adomian decomposition method (ADM) and the homotopy perturbation method (HPM) are among applicable methods for solving various problems such as the Klein-Gordon equation, Triki-Biswas equation, the problem of boundary layer convective heat transfer, integral equations (IEs) of the first and second kinds with hypersingular kernels, the Volterra integral form of the Lane-Emden equations with initial values and boundary conditions, Cauchy IEs of the first kind, linear and nonlinear IEs and partial differential equations.

In the iterative schemes based on the FPA, usually the numerical results are obtained for special iteration or the following termination criterion is applied:

$$|y(s) - y_j(s)| \leq \epsilon, \tag{1}$$

where the value ϵ is a given tolerance and $y_j(s)$ is the approximate solution of $y(s)$.

Let y_j for $j \geq 1$ be the approximate solution our problem which are produced by one of the numerical methods. In Eq. (1) which is applied in the FPA, for ϵ enough large, the suitable approximation can not be obtained and for small values of ϵ the unnecessary iterations can be produced.

In the SA, instead of Eq. (1), the following termination criterion is replaced:

$$|y_j(s) - y_{j+1}(s)| = @.0, \tag{2}$$

where the sign @.0 is the informatical zero in the SA.

Also, recently this method has been applied to validate the results of the Newton-cotes integration rule, Gaussian integration rule, collocation method for solving Fredholm IEs, finding the optimal convergence control parameter of the homotopy analysis method, solving fuzzy IEs by Sinc-collocation method, solving fuzzy numerical integrals, finding the optimal regularization parameter for solving first kind IEs, solving load leveling problem and solving the VIEs with discontinuous kernel using the Taylor-collocation method.

Stochastic Arithmetic and the CESTAC method

If we produce the representable values by computer and collect them in B, then $S^* \in B$ can be written for $s^* \in \mathbb{R}$ with α mantissa bits of the binary FPA as

$$S^* = s^* - \rho 2^{E-\alpha}\phi, \tag{3}$$

where $\rho, 2^{-\alpha}\phi$ and E are sign, missing segment of the mantissa and the binary exponent of the result respectively. Also, we know that for $\alpha = 24, 53$, the numerical results can be produced in single and double precisions.

By assuming ϕ as a casual variable that uniformly distributed on $[-1, 1]$, we will be able to make perturbation on last mantissa bit of s^*. Then the mean (μ) and the standard deviation (σ) values can be produced for results of S^* which have important role to identify the precision of S^*. If we repeat the process for k times, we will have the quasi Gaussian distribution on $S_i^*, i = 1, \cdots, k$ and we will have equality between μ and the exact s^*.
Algorithm 1, shows the process step by step where τ_δ is the value of T distribution as the confidence interval is $1 - \delta$ with $k - 1$ freedom degree.

Algorithm 1:

Step 1- Produce k samples of S^* in the form of $\Phi = \{S_1^*, S_2^*, ..., S_k^*\}$ by making perturbation on the last bit of mantissa.

Step 2- Calculate $\tilde{S}^* = \dfrac{\sum_{i=1}^{k} S_i^*}{k}$.

Step 3- Find $\sigma^2 = \dfrac{\sum_{i=1}^{k}(S_i^* - \tilde{S}^*)^2}{k-1}$.

Step 4- Apply $C_{\tilde{S}^*, S^*} = \log_{10} \dfrac{\sqrt{k}\left|\tilde{S}^*\right|}{\tau_\delta \sigma}$ to find the NCSDs between S^* and $\tilde{S}^*$.

Step 5- Show $S^* = @.0$ if $\tilde{S}^* = 0$, or $C_{\tilde{S}^*, S^*} \leq 0$.

The Advantages of Using the CESTAC

The applying of the CESTAC or perturbation method in a scientific program has the following advantages:

- The accuracy of any numerical result is estimated, during the running of a program.
- The numerical instabilities are detected and the branching are checked.

- Unnecessary iterations are eliminated which the floating-point arithmetic is not able to distinguish them. In some cases, the termination criterion of iterative methods is not suitable so that, the implementation of the algorithm is continued without improvement in the accuracy of the result. In the stochastic arithmetic, instead of the termination criterion, a criterion that directly reflects the mathematical condition, is replaced, that must be satisfied by the solution.
- It is able to find the optimal step of the iterative methods, which after this step, the accuracy of the result does not increases or maybe decreases, because of the rounding error accumulation.
- It is an effective and powerful tool that helps to achieve the validation of scientific programs and gives them a reliability.

The CADNA library

In order to implement the DSA, the CESTAC method must be used. This method is able to convert the FPA to the DSA based on a probabilistic approach. CADNA is a library which was designated by Chesneaux in order to perform the CESTAC method on a code written by C++ or FORTRAN automatically.

This study applies the ADM for solving the linear and non-linear VIE with discontinuous kernel and validates the numerical results using the CESTAC method and the CADNA library. So we will be able to find the optimal approximation, the optimal error and the optimal iteration of the ADM for solving Eq. (20). The uniqueness theorem, the error theorem and the convergence theorem of the ADM are proved. Also, the main theorem of the CESTAC method is discussed. Based on this theorem, we can apply the new termination criterion instead of the absolute error. Several examples are solved and the CESTAC method is applied to validate the results and finding the optimal results of the ADM for solving the mentioned problem.

Adomian Decomposition method

Consider the following second kind nonlinear VIE with discontinuous kernel

$$y(t) = x(t)+\int_{\beta_0(t)}^{\beta_1(t)} k_1(t,\tau)F(y(\tau))d\tau+\int_{\beta_1(t)}^{\beta_2(t)} k_2(t,\tau)F(y(\tau))d\tau+\cdots+\int_{\beta_{m'-1}(t)}^{\beta_{m'}(t)} k_{m'}(t,\tau)F(y$$

such that it can be rewritten as follows

$$y(t) = x(t) + \sum_{j=1}^{m'} \int_{\beta_{j-1}(t)}^{\beta_j(t)} k_j(t,\tau)F(y(\tau))d\tau, \qquad (4)$$

where $\forall t \in J = [0, T]$ we assume that $x(t)$ is bounded and $k_j(t,\tau)$ is discontinuous along continuous curves $\beta_j(t),\ j = 0, 1, \cdots, m$ such that $|k_j(t,\tau)| < M_j,\ \forall 0 \leq \tau \leq t \leq T$.

Let the nonlinear term $F(y)$ satisfies in the Lipschitz continuous such that $|F(y) - F(z)| \leq L|y - z|$ and we can write the Adomian polynomial representation for $F(y)$ as follows

$$F(y) = \sum_{n=0}^{\infty} A_n(y_0, y_1, \cdots, y_n), \qquad (5)$$

where we have

$$A_n = \left(\frac{1}{n!}\right)\left(\frac{d^n}{d\lambda^n}\right)\left[f\left(\sum_{n=0}^{\infty} \lambda^i y_i\right)\right]_{\lambda=0}. \qquad (6)$$

Also, we know that the Adomian polynomials can be shown in the following form:

$$A_n = F(P_n) - \sum_{j=0}^{n-1} A_j, \tag{7}$$

where $P_n = \sum_{i=0}^{n} y_i(t)$ shows the partial sum. The solution of Eq. (20) applying the ADM can be obtained using

$$y(t) = \sum_{i=0}^{\infty} y_i(t), \tag{8}$$

where

$$y_0(t) = x(t),$$

$$y_i(t) = \sum^{m'} \int_{\beta_{j-1}(t)}^{\beta_j(t)} k_j(t,\tau) A_{i-1} d\tau, \ i \geq 1. \tag{9}$$

Theorem

If we apply the ADM for solving Eq. (20), the obtained solution will be unique whenever $0 < \eta < 1$ *where* $\eta = L \sum_{j=1}^{m'} M_j(\beta_j - \beta_{j-1})$.

Theorem

The series solution (24) for solving Eq. (20) using the ADM converges if $0 < \eta < 1$ *and* $|y_1| < \infty$

Theorem

If we apply the series solution (24) for solving Eq. (20), the maximum absolute error truncation can be obtained as follows

$$\max \left| y(t) - \sum_{i=0}^{m} y_i(t) \right| \leq \frac{k\eta^{m+1}}{L(1-\eta)}$$

where $k = \max_{\forall t \in J} |F(x(t))|$.

Definition

The NCSDs for two real numbers r_1, r_2 can be obtained as follows
(1) for $r_1 \neq r_2$,

$$C_{r_1,r_2} = \log_{10}\left|\frac{r_1 + r_2}{2(r_1 - r_2)}\right| = \log_{10}\left|\frac{r_1}{r_1 - r_2} - \frac{1}{2}\right|, \qquad (10)$$

(2) for all real numbers r_1, $C_{r_1,r_1} = +\infty$.

Theorem

Let $y(t)$ and $y_n(t)$ be the exact and numerical solutions of problem (20) which $y_n(t)$ is obtained by using the ADM. We have

$$C_{y_n(t),y_{n+1}(t)} \simeq C_{y_n(t),y(t)}, \tag{11}$$

where $C_{y_n(t),y(t)}$ shows the NCSDs of $y_n(t), w(t)$ and $C_{y_n(t),y_{n+1}(t)}$ is the NCSDs of two successive iterations $y_n(t), y_{n+1}(t)$.

Homotopy Perturbation Method

Let for operator $\mathcal{F}$, given function g and prepare function x we get the following operator equation as

$$\mathcal{F}(x) = g(z), \quad z \in \Gamma. \tag{12}$$

We can write the operator $\mathcal{F}$ in the following form

$$\mathcal{F}(x) = \mathcal{L}(x) + \mathcal{N}(x), \tag{13}$$

where the remain part of $\mathcal{F}$ showed by $\mathcal{N}$ and $\mathcal{L}$ is the linear operator. Now, Eq. (12) can be presented as

$$\mathcal{L}(x) + \mathcal{N}(x) = g(z), \quad z \in \Gamma. \tag{14}$$

According to the traditional homotopy, for parameter $\hat{a} \in [0, 1]$, the homotopy operator H can be presented as

$$H(v, \hat{a}) = (1 - \hat{a})(\mathcal{L}(v) - \mathcal{L}(x_0)) + \hat{a}(\digamma(v) - g(z)), \qquad (15)$$

where $v(z, \hat{a})$ is defined on $\Gamma \times [0, 1] \to R$ and x_0 is the prime guess of Eq. (12). Now, by applying Eq. (13) we get

$$H(v, \hat{a}) = \mathcal{L}(v) - \mathcal{L}(x_0) + \hat{a}\mathcal{L}(x_0) + \hat{a}(\mathcal{N}(v) - g(z)). \qquad (16)$$

Putting $\hat{a} = 0$ in Eq. (16) leads to $H(v, 0) = \mathcal{L}(v) - \mathcal{L}(x_0)$ and we get $\mathcal{L}(v) - \mathcal{L}(x_0) = 0$. Now, for $\hat{a} = 1$ we have $H(v, 1) = 0$ which it can produce the solution of Eq. (12). Thus, when $\hat{a} : 0 \to 1$ we can change the solution v from x_0 to x. Now, the power series

$$v = \sum_{j=0}^{\infty} \hat{a}^j v_j, \tag{17}$$

can be applied to find the solution of $H(v, \hat{a}) = 0$. Then comparing the same powers of parameter $\hat{a}$ we can find the successive functions $v_j, j = 0, \cdots, n$.

Finally, applying

$$w = \lim_{\hat{a} \to 1} v = \sum_{j=0}^{\infty} v_j, \tag{18}$$

the solution of Eq. (12) can be found and the n-th order approximation is in the following form

$$w_n = \sum_{j=0}^{n} v_j. \tag{19}$$

The HPM for solving first kind Volterra IEs

Consider the following first kind linear IE

$$\int_0^t k(t,s)v(s)ds = f(t), a = 0 \le t \le T \le b, \tag{20}$$

where $k(t,s)$ is discontinuous along continuous curves $\gamma_i, i = 0, 1, \cdots, m-1$. So Eq. (20) can be written in the following form

$$\int_{\gamma_0(t)}^{\gamma_1(t)} k_1(t,s)v(s)ds + \int_{\gamma_1(t)}^{\gamma_2(t)} k_2(t,s)v(s)ds + \cdots + \int_{\gamma_{m-1}(t)}^{\gamma_m(t)} k_m(t,s)v(s)ds \tag{21}$$

and finally for linear form we get

$$\sum^{m} \int^{\gamma_i(t)} k_i(t,s)v(s)ds = f(t). \tag{22}$$

Also, the nonlinear first kind Volterra IE with jumping kernel is in the following form

$$\sum_{i=1}^{m}\int_{\gamma_{i-1}(t)}^{\gamma_i(t)} k_i(t,s)F[v(s)]ds = f(t). \tag{23}$$

For solving IE (23), using the transformation

$$w(s) = F[v(s)], \tag{24}$$

we can convert the nonlinear Eq. (23) to the linear form as follows

$$\sum_{i=1}^{m}\int_{\gamma_{i-1}(t)}^{\gamma_i(t)} k_i(t,s)w(s)ds = f(t). \tag{25}$$

Disjoint the first term of Eq. (25) and by adding and subtracting $w(t)$ to the second part of obtained relation we have

$$\int_{\gamma_0(t)}^{\gamma_1(t)} k_1(t,s)w(s)ds+\sum_{i=2}^{m}\int_{\gamma_{i-1}(t)}^{\gamma_i(t)} k_i(t,s)[w(s)+w(t)-w(t)]ds = f(t). \tag{26}$$

Now, we can write

$$w(t)\sum_{i=2}^{m}\int_{\gamma_{i-1}(t)}^{\gamma_i(t)} k_i(t,s)ds$$

$$= f(t) - \int_{\gamma_0(t)}^{\gamma_1(t)} k_1(t,s)w(s)ds - \sum_{i=2}^{m}\int_{\gamma_{i-1}(t)}^{\gamma_i(t)} k_i(t,s)[w(s)-w(t)]ds, \tag{27}$$

and dividing both sides of above equation to $H_1(t) = \sum_{i=2}^{m} \int_{\gamma_{i-1}(t)}^{\gamma_i(t)} k_i(t,s)ds$ the following second kind integral equation can be obtained as

$$w(t) = \frac{f(t)}{H1(t)}$$

$$-\frac{1}{H1(t)} \left[\int_{\gamma_0(t)}^{\gamma_1(t)} k_1(t,s)w(s)ds - \sum_{i=2}^{m} \int_{\gamma_{i-1}(t)}^{\gamma_i(t)} k_i(t,s)[w(s) - w(t)]ds \right]. \quad (28)$$

Thus, Eq. (28) can be written in the following form

$$w(t) = g(t) - \int_{\gamma_0(t)}^{\gamma_1(t)} k_1'(t,s)w(s)ds - \sum_{i=2}^{m} \int_{\gamma_{i-1}(t)}^{\gamma_i(t)} k_i'(t,s)[w(s) - w(t)]ds \tag{29}$$

where $g(t) = \frac{f(t)}{\sum_{i=2}^{m} \int_{\gamma_{i-1}(t)}^{\gamma_i(t)} k_i(t,s)ds}$ and $k_i'(t,s) = \frac{k_i(t,s)}{\sum_{i=2}^{m} \int_{\gamma_{i-1}(t)}^{\gamma_i(t)} k_i(t,s)ds}$.

Based on the HPM and applying Eqs. (13) and (14) for solving Eq. (29), the linear and non-linear operators $\mathcal{L}(v)$ and $\mathcal{N}(v)$ should be defined as follows

$$\mathcal{L}(v) = v,$$

and

$$\mathcal{N}(v) = \int_{\gamma_0(t)}^{\gamma_1(t)} k_1'(t,s)w(s)ds - \sum_{i=2}^{m} \int_{\gamma_{i-1}(t)}^{\gamma_i(t)} k_i'(t,s)[w(s)-w(t)]ds. \tag{30}$$

For next step, using Eq. (16) the homotopy map can be constructed as follows

$$H(v,\hat{a}) = v(t) - w_0(t)$$

$$+\hat{a}[w_0(t) + \int_{\gamma_0(t)}^{\gamma_1(t)} k_1'(t,s)w(s)ds + \sum_{i=2}^{m} \int_{\gamma_{i-1}(t)}^{\gamma_i(t)} k_i'(t,s)[w(s) - w(t)]ds \quad (31)$$

and we have

$$\sum_{j=0}^{\infty} \hat{a}^j v_j(t) = w_0(t)$$

$$+\hat{a}[g(t) - w_0(t)] - \sum_{j=1}^{\infty} \hat{a}^j \int_{\gamma_0(t)}^{\gamma_1(t)} k_1'(t,s) v_{j-1}(s) ds \qquad (32)$$

$$-\sum_{j=1}^{\infty} \hat{a}^j \sum_{i=2}^{m} \int_{\gamma_{i-1}(t)}^{\gamma_i(t)} k_i'(t,s)[v_{j-1}(s) - v_{j-1}(t)] ds.$$

Now, Eq. (32) can be written in the following form

$$\sum_{j=0}^{\infty} \hat{a}^j v_j(t) = w_0(t)+\hat{a}[g(t)-w_0(t)]-\sum_{j=1}^{\infty} \hat{a}^j A_{j-1}(t)-\sum_{j=1}^{\infty} \hat{a}^j B_{j-1}(t), \tag{33}$$

where

$$A_{j-1}(t) = \int_{\gamma_0(t)}^{\gamma_1(t)} k_1'(t,s)v_{j-1}(s)ds,$$

and

$$B_{j-1}(t) = \sum_{i=2}^{m} \int_{\gamma_{i-1}(t)}^{\gamma_i(t)} k_i'(t,s)[v_{j-1}(s) - v_{j-1}(t)]ds.$$

By disjointing the different powers of $\hat{a}$ in both sides of Eq. (33) the following successive iterations can be obtained as

$$\hat{a}^0 : v_0(t) = w_0(t),$$

$$\hat{a}^1 : v_1(t) = g(t) - w_0(t) - A_0(t) - B_0(t)$$

$$= g(t) - w_0(t) - \int_{\gamma_0(t)}^{\gamma_1(t)} k_1'(t,s)v_0(s)ds - \sum_{i=2}^{m} \int_{\gamma_{i-1}(t)}^{\gamma_i(t)} k_i'(t,s)[v_0(s) - v_0(t)]ds,$$

$$\hat{a}^2 : v_2(t) = -A_1(t) - B_1(t) = -\int_{\gamma_0(t)}^{\gamma_1(t)} k_1'(t,s)v_1(s)ds - \sum_{i=2}^{m} \int_{\gamma_{i-1}(t)}^{\gamma_i(t)} k_i'(t,s)[v_1(s) - v_1(t)]d$$

$$\vdots \quad \vdots \qquad \vdots$$

$$\hat{a}^n : v_n(t) = -A_{n-1}(t) - B_{n-1}(t) = -\int_{\gamma_0(t)}^{\gamma_1(t)} k_1'(t,s)v_{n-1}(s)ds - \sum_{i=2}^{m} \int_{\gamma_{i-1}(t)}^{\gamma_i(t)} k_i'(t,s)[v_{n-1}(s$$

(34)

Applying Eq. (19) and successive iterations (34), the approximate solution of Eq. (29) can be obtained. In order to find the approximate solution of (23) we need to inverse of Eq. (24) as $v(s) = F^{-1}[w(s)]$.

Theorem

Assume that functions $k_i'(t,s)$ and $g(t)$ of Eq. (29) are continuous in $\eta_1 = [a,b] \times [a,b]$ and $\eta = [a,b]$ respectively where these functions are bounded. It shows that

$$\exists \alpha_i, N_1; |k_i'(t,s)| \le \alpha_i, |g(t)| \le N_1, \ \forall s,t \in [a,b], \ i = 1,2,\cdots,m,$$

then for initial approximation w_0 which is continuous in $[a,b]$, series solution (18) will be uniformly convergent to the exact solution for each $\hat{a} \in [0,1]$.

Theorem

Based on the n-th order approximate solution (19), the error function $E_n = \sup_{t\in[a,b]} |w(t) - w_n(t)|$ *can be approximated as follows*

$$E_n \leq \beta \left[\exp\left(\alpha_1(\gamma_1 - \gamma_0)\right) + \exp\left(2\sum_{i=2}^{m} \alpha_i(\gamma_i - \gamma_{i-1})\right) - \sum_{j=0}^{n-1}\left(\alpha_1^j \frac{(\gamma_1 - \gamma_0)^j}{j!} + 2^j \sum_{i=2}^{m} \alpha_i^j \frac{(\gamma_i - \gamma_{i-1})^j}{j!}\right)\right].$$

Theorem

Let $w(t)$ and $w_n(t)$ be the exact and numerical solutions of problem (29) which $w_n(t)$ is obtained by using the HPM and Eq. (19). Based on assumptions of theorems 1 and 2 for n enough large we have

$$C_{w_n(t),w_{n+1}(t)} \simeq C_{w_n(t),w(t)}, \tag{35}$$

where $C_{w_n(t),w(t)}$ shows the NCSDs of $w_n(t), w(t)$ and $C_{w_n(t),w_{n+1}(t)}$ is the NCSDs of two successive iterations $w_n(t), w_{n+1}(t)$.

Numerical Results of the ADM

Example 1: Consider the following linear VIE with discontinuous kernel

$$y(t) = x(t) + \int_0^{\frac{t}{8}} 2ty(\tau)d\tau + \int_{\frac{t}{8}}^{\frac{3t}{8}} (t-\tau)y(\tau)d\tau + \int_{\frac{3t}{8}}^{t} y(\tau)d\tau,$$

where

$$x(t) = -\cos(\frac{3t}{8}) + 2\cos(t) - 2t\left(1 - \cos(\frac{t}{8}) + \sin(\frac{t}{8})\right)$$
$$+\frac{1}{8}(-(8+7t)\cos(\frac{t}{8}) + (8+5t)\cos(\frac{3t}{8})$$
$$-2\left(-t + (-8+5t)\cos(\frac{t}{4})\right)\sin(\frac{t}{8})) + \sin(\frac{3t}{8}),$$

and the exact solution is $y(t) = \sin t + \cos t$.

Table: The numerical results of Example 1 for $\varepsilon = 10^{-5}$ based on the FPA.

n	$y_{n+1}(t)$	$\lvert y_{n+1}(t) - y(t)\rvert$
1	0.98828512430191040039	0.26257163286209106445
2	1.21838188171386718750	0.03247487545013427734
3	1.24792301654815673828	0.00293374061584472656
4	1.25064921379089355469	0.00020754337310791016
5	1.25084471702575683594	0.00001204013824462891
6	1.25085616111755371094	0.00000059604644775391

Table: The number of iterations for different values of ε based on the FPA.

ε	Small values	$\varepsilon = 10^{-5}$	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-1}$	$\varepsilon = 0.5$	Large values
n	$>> 6$	6	4	2	1	1

Table: The numerical results of Example 1 based on the CESTAC method.

n	$y_{n+1}(t)$	$\lvert y_{n+1}(t)-y_n(t)\rvert$	$\lvert y_{n+1}(t)-y(t)\rvert$
1	0.988285E+000	0.988285E+000	0.262571E+000
2	0.121838E+001	0.230096E+000	0.32474E-001
3	0.124792E+001	0.2954E-001	0.293E-002
4	0.125064E+001	0.272E-002	0.20E-003
5	0.125084E+001	0.195E-003	0.1E-004
6	0.125085E+001	0.1E-004	@.0
7	0.125085E+001	@.0	@.0

Example 2: Consider the following nonlinear VIE with non-smooth kernel

$$y(t) = x(t) + \int_0^{\frac{t}{2}} (t-\tau) y^2(\tau) d\tau + 2 \int_{\frac{t}{2}}^{t} y^2(\tau) d\tau,$$

where

$$x(t) = \sin(t) + \frac{1}{16}(2 - 3t^2 - 2\cos(t) + 2t\sin(t)) + \frac{1}{2}(-t - \sin(t) + 2\cos(t)\sin$$

and the exact solution is $y(t) = \sin t$.

Table: The numerical results using the CESTAC method and the CADNA library.

n	$y_{n+1}(t)$	$\lvert y_{n+1}(t) - y_n(t)\rvert$	$\lvert y_{n+1}(t) - y(t)\rvert$
1	0.194002E+000	0.194002E+000	0.46670E-002
2	0.198526E+000	0.45239E-002	0.143E-003
3	0.198798E+000	0.272E-003	0.129E-003
4	0.198819E+000	0.20E-004	0.150E-003
5	0.198821E+000	0.1E-005	0.15E-004
6	0.198821E+000	@.0	@.0

Table: The number of iterations for different values of ε based on the FPA.

ε	Small values	$\varepsilon = 10^{-5}$	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-1}$	$\varepsilon = 0.5$	Large values
n	$>> 8$	8	2	1	1	1

Numerical results of the HPM

Example

Consider the following nonlinear Volterra IE of the first kind

$$\int_0^{\frac{t}{4}}(s^2+t-1)v^3(s)ds+\int_{\frac{t}{4}}^{\frac{t}{2}}(s-t)v^3(s)ds+\int_{\frac{t}{2}}^{t}tv^3(s)ds$$

$$=-\frac{1}{114688}t^7+\frac{517945}{3670016}t^8+\frac{1}{2359296}t^9, \tag{36}$$

where the exact solution is $v(t)=t^2$.

Applying transformation (24) we can change the nonlinear terms to the linear form, then using the mentioned technique we can change the first kind IE (36) to the second kind

$$w(t) = g(t) - \frac{1}{H_1(t)} \int_0^{\frac{t}{4}} (s^2 + t - 1) w(s) ds$$

$$- \frac{1}{H_1(t)} \int_{\frac{t}{4}}^{\frac{t}{2}} (s - t) w(s) ds - \frac{1}{H_1(t)} \int_{\frac{t}{2}}^{t} t w(s) ds$$

where $H_1(t) = \int_{\frac{t}{4}}^{\frac{t}{2}} (s-t)ds + \int_{\frac{t}{2}}^{t} tds = \frac{11t^2}{33}$ and $g(t) = \frac{f(t)}{H_1(t)} = \frac{t^5(-288+4661505t+14t^2)}{11354112}$. In this example, the SA and the CADNA library are applied to calculate the numerical approximations. Also, using the stopping condition (2) we do not need to have the exact solution to show that accuracy of presented method. The numerical results are presented in Table 6 for $t = 0.5$. Using this results, the optimal approximation is $0.2499987E + 000$ and the optimal absolute error is $0.12E - 005$ and $n = 21$ is the optimal step of iteration for HPM method for solving Example 9.

Table: Applying Algorithm 2 for Example 9 with $t = 0.5$.

n	$w_n(t)$	$\lvert w_n(t) - w_{n+1}(t)\rvert$	$\lvert w(t) - w_n(t)\rvert$
1	0.2168369E+000	0.2168369E+000	0.331630E-001
2	0.2316099E+000	0.147729E-001	0.183900E-001
3	0.2395026E+000	0.7892757E-002	0.104973E-001
4	0.2439217E+000	0.44190E-002	0.6078287E-002
5	0.2464545E+000	0.25328E-002	0.35454E-002
6	0.2479224E+000	0.14678E-002	0.20775E-002
7	0.2487795E+000	0.85714E-003	0.12204E-002
8	0.2492820E+000	0.5024E-003	0.71795E-003
9	0.2495773E+000	0.2952E-003	0.4226E-003
10	0.2497509E+000	0.1736134E-003	0.2490E-003
11	0.2498532E+000	0.102E-003	0.1467E-003
12	0.2499134E+000	0.602E-004	0.8650E-004
13	0.2499490E+000	0.355E-004	0.509E-004
14	0.2499699E+000	0.2093613E-004	0.3002583E-004
15	0.2499823E+000	0.123E-004	0.176E-004
16	0.2499895E+000	0.724E-005	0.10E-004
17	0.2499938E+000	0.42E-005	0.61E-005
18	0.2499964E+000	0.25E-005	0.35E-005
19	0.2499978E+000	0.14E-005	0.21E-005
20	0.2499987E+000	0.9E-006	0.12E-005
21	0.2499987E+000	@.0	@.0

Example

Consider the following nonlinear first kind Volterra IE with discontinuous kernel

$$\int_0^{\frac{t}{3}} (s+t)v^2(s)ds + \int_{\frac{t}{3}}^{t} v^2(s)ds = \frac{-2t}{3} + \frac{t^2}{2} + \frac{11t^3}{81}, \qquad (37)$$

with non-smooth solution $v(t) = \sqrt{2t-1}$. For solving this example, the following algorithm will be applied.

In this example, in order to show the accuracy of method, the CESTAC method and the CADNA library are applied. Also, instead of applying the termination criterion (1) and using the traditional absolute error the novel stopping condition (2) is applied. This condition is based on the two successive approximations $w_n(t)$ and $w_{n+1}(t)$. When the difference of these terms is @.0 the CESTAC algorithm will be stopped. It shows that the NCSDs between two successive iterations is zero. The numerical results using the SA are presented in Table 7 for $t = 0.7$. According to this table the optimal step of iterations for HPM is $n = 15$, the optimal approximation is $0.6324449E + 000$ and the optimal absolute error is $0.10E - 004$.

Table: Applying Algorithm 2 for Example 10 with $t = 0.7$.

n	$w_n(t)$	$\lvert w_n(t) - w_{n+1}(t)\rvert$	$\lvert w(t) - w_n(t)\rvert$
1	0.469950E+000	0.469950E+000	0.162504E+000
2	0.6163554E+000	0.146404E+000	0.16100E-001
3	0.6425781E+000	0.26222E-001	0.10122E-001
4	0.6433458E+000	0.7676E-003	0.10890E-001
5	0.6397220E+000	0.3623E-002	0.72665E-002
6	0.6365918E+000	0.31302E-002	0.4136E-002
7	0.634611E+000	0.1980E-002	0.2156E-002
8	0.633508E+000	0.1103E-002	0.1052E-002
9	0.6329386E+000	0.569E-003	0.483E-003
10	0.6326575E+000	0.2810359E-003	0.202E-003
11	0.632525E+000	0.131E-003	0.70E-004
12	0.6324732E+000	0.52E-004	0.17E-004
13	0.6324512E+000	0.21E-004	0.42E-005
14	0.6324449E+000	0.62E-005	0.10E-004
15	0.6324449E+000	@.0	0.10E-004

Example

Consider the following linear Volterra IE of the first kind

$$\int_0^{\frac{t}{8}} (t+s^2)v(s)ds + \int_{\frac{t}{8}}^{\frac{3t}{8}} v(s)ds - 3\int_{\frac{3t}{8}}^{t} v(s)ds$$
$$= -\frac{13t}{8} + \frac{t^2}{8} + \frac{t^3}{1536} - \frac{11965t^4}{16384} + \frac{t^5}{16384} + \frac{t^6}{1572864}, \tag{38}$$

where the exact solution is $v(t) = t^3 + 1$. For solving this example, we need to repeat the mentioned process in previous examples.

Thus instead of solving the first kind IE (38) the following second kind IE

$$w(t) = g(t) - \frac{1}{H_1(t)} \int_0^{\frac{t}{8}} (t + s^2) v(s) ds$$

$$- \frac{1}{H_1(t)} \int_{\frac{t}{8}}^{\frac{3t}{8}} v(s) ds + \frac{1}{H_1(t)} \int_{\frac{3t}{8}}^{t} 3v(s) ds$$

should be solved where $H_1(t) = \int_{\frac{t}{8}}^{\frac{3t}{8}} ds - 3 \int_{\frac{3t}{8}}^{t} ds = \frac{-13t}{8}$ and $g(t) = \frac{f(t)}{H_1(t)} = 1 - \frac{t}{13} - \frac{t^2}{2496} + \frac{11965t^3}{26624} - \frac{t^4}{26624} - \frac{t^5}{2555904}$. The numerical results are presented in Table 8. The optimal iteration of the HPM for solving this example is $n = 11$, the optimal approximation is $0.100799E + 001$ and the optimal error is $0.7E - 005$.

Table: Numerical approximations for Example 11 with $t = 0.2$.

n	$w_n(t)$	$\|w_n(t) - w_{n+1}(t)\|$	$\|w(t) - w_n(t)\|$
0	0.1001780E+001	0.1001780E+001	0.6219267E-002
1	0.1005726E+001	0.3945E-002	0.2273E-002
2	0.100703E+001	0.130E-002	0.968E-003
3	0.1007536E+001	0.504E-003	0.463E-003
4	0.100776E+001	0.22E-003	0.238E-003
5	0.100787E+001	0.11E-003	0.127E-003
6	0.100793E+001	0.58E-004	0.68E-004
7	0.100796E+001	0.3E-004	0.38E-004
8	0.100797E+001	0.16E-004	0.2E-004
9	0.100798E+001	0.9E-005	0.1E-004
10	0.100799E+001	0.4E-005	0.7E-005
11	0.100799E+001	@.0	@.0

Conclusion

Comparing the results we can see some advantages of the CESTAC method, CADNA library and SA in comparison with FPA and usual mathematical packages.

Thank you for your attention!!

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