LAVRENTYEV INSTITUTE OF HYDRODYNAMICS OF SB RAS

MATHEMATICAL CENTER IN AKADEMGORODOK

THE THIRD RUSSIA-JAPAN WORKSHOP

MATHEMATICAL ANALYSIS OF FRACTURE PHENOMENA FOR ELASTIC STRUCTURES AND ITS APPLICATIONS

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ABSTRACTS

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The mathematical foundation of fracture mechanics has seen considerable advances in the last years. This field of study covers a big variety of exciting topics, including propagation of cracks, equilibrium of structures with thin inclusions in the presence of delaminations, frictional contact problems, inverse and control problems. The aim of the workshop ”Mathematical analysis of fracture phenomena for elastic structures and its applications” is to bring together researchers working on different aspects of these issues. The workshop provides a platform for researchers to communicate, discuss, and exchange ideas under the common theme of fracture phenomena.

The first Workshop Mathematical analysis of fracture phenomena for elastic structures and its applications´ was hosted in Novosibirsk in November 2019 due to the collaboration of researchers from Japan and Russia. The Second Russia-Japan Workshop was held in December 2020. The geography of the workshop participants was expanded: researchers from Russia, Japan, Germany, Austria and Czech Republic were involved. In 2020, the Workshop was integrated with the 20th Conference of Continuum Mechanics Focusing on Singularities (CoMFoS20).

CoMFoS was initiated in 1995 under the auspices of the activity group ”Continuum Mechanics Focusing on Singularities (CoMFoS)” of the Japan Society for Industrial and Applied Mathematics (JSIAM). From April 2010, the activity group CoMFoS was renamed ”Mathematical Aspects of Continuum Mechanics (MACM)”. This is the 21st ´onference of CoMFoS and will be held under the co-sponsorship of the Japan - Russia Research Cooperative Program.

The Workshop and CoMFoS topics:

- elasticity, plasticity
- modeling of composite materials
- fracture mechanics
- study of mathematical models for solids with defects
- asymptotic and multiscale analysis
- optimal shape design
- inverse problems

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Contents

LOCAL WELL-POSEDNESS FOR SOME PHASE-FIELD MODEL OF COMPLETE DAMAGE

Goro Akagi

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This talk is concerned with local (in time) well-posedness for Frémond's model of complete damage in elastic materials (see, e.g., [2, 3]). Frémond's model is a phase-field model for damage and consists of elliptic and parabolic equations for a displacement field $u = u(x,t)$ and a phase-field $z = z(x,t)$, which represents the locally averaged evolution of damage at any point $x \in \Omega$ and $t \in (0,T)$ (in particular, $z = 1$ when the material is completely integer and $z = 0$ when the elastic bonds have been broken) and which is supposed to be monotone in time due to the irreversible feature of damage. Similar phase-field models for damage and fracture have been proposed and studied vigorously so far; however, well-posedness of such phase-field models often remain widely open for many years due to severe nonlinearity intrinsic to the models. In this talk, we shall discuss local well-posedness for the following system:

$$\begin{aligned} -\mathrm{div}(z\nabla u) &= g && \text{in } \Omega \times (0,T), \\ \partial I_{(-\infty,0]}(z_t) + z_t - \Delta z + \psi'(z) &\ni -\frac{1}{2}|\nabla u|^2 && \text{in } \Omega \times (0,T), \end{aligned}$$

where $g = g(x)$ is a given data, $\partial I_{(-\infty,0]}$ denotes the subdifferential operator of the indicator function $I_{(-\infty,0]}$ supported over $(-\infty, 0]$ and ψ is a smooth λ-convex potential, equipped with certain boundary and initial conditions. It is noteworthy that the elliptic constant of the first equation vanishes when $z(x,t)$ is equal to zero (i.e., the material is completely broken at (x,t)), and therefore, this system is called a *complete damage model* and one may expect only local (in time) well-posedness. This talk is based on a joint work [1] with Giulio Schimperna (Pavia, Italy).

REFERENCES

1. Akagi, G. and Schimperna, G., *Local well-posedness for Frémond's model of complete damage in elastic solids.* to appear in Euro. J. Appl. Math.
2. Frémond, M., *Non-smooth Thermomechanics.* Springer, Berlin, 2002.
3. Frémond, M., *Phase Change in Mechanics.* Springer-Verlag, Berlin Heidelberg, 2012.

PRINCIPLES OF ULTRASONIC DRILLING OF EXTRATERRESTRIAL OBJECTS: THEORETICAL JUSTIFICATION AND PRACTICAL IMPLEMENTATION

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Drilling using ultrasonic vibrations is the most promising way to study extraterrestrial objects due to the possibility of significantly increasing the rate of the channels making with ensuring minimal destruction of the structure and content of the treated soils. Since the task of ensuring the maximum rate of the channels making to a given depth is being solved, for example, for fixing landing modules on asteroids, it becomes necessary to determine the properties of the treated soil in order to setup optimal modes of ultrasonic (or combined with ultrasonic) exposure. When drilling on planets, the task of studying the properties of soils, detecting valuable substances (water) in the deep layers of the soil without taking them and transporting them to research compartments or to Earth arises.

To solve the problem, a method for monitoring the properties of the soil of extraterrestrial objects during ultrasonic drilling has been proposed and is being developed.

The principles of the process implementation based on the effects in the following sequence and combinations are proposed:

- preliminary low-amplitude exposure (with a range of the working end of the emitter of no more than 10 microns) for indirect measurement of the mechanical properties of the soil by mechanical impedance proportional to the complex resistance of the mechanical branch of the equivalent electrical circuit of the piezoelectric ultrasonic oscillatory system;
- high-amplitude exposure with automatically set maximum effective modes of ultrasonic exposure simultaneously with low-frequency shocks and/or pseudo-rotation, depending on the type and properties of the soil.

For theoretical substantiation and identification of optimal modes of each stage of impact, models for determining the relationship between the impedance and mechanical properties of the soil and determining the drilling speed are proposed and developed. The peculiarity of the model is to take into account the impact-contact effect, i.e. periodic separation of the radiator from the soil surface. The analysis of the model based on the linear theory of elasticity allowed us to determine the relationship between the impedance and the mechanical properties of the soil. Numerical analysis of the influence of the mechanical properties of the soil on the impedance allowed us to determine that it is most expedient to control the cosine of the phase shift angle between the force and the displacement velocity of the working end of the radiator relative to the input end (with a swing of 6 microns, the cosine of the phase shift angle decreases by almost 2 times with an increase in the elastic modulus from $1 \cdot 10^{10}$ to $5 \cdot 10^{10}$ Pa).

The proposed theory of soil destruction under the influence of ultrasonic vibrations is based on the model of K.B. Broberg describing the development and coalescence of microcracks, provided that the accumulation of microcracks before the destruction of the material occurs when exposed for several periods of vibrations.

To simplify calculations with primary high-frequency oscillatory action, a method for reducing the destruction problem to an approximate system of ordinary differential equations of the 2nd order was proposed.

The theoretical studies made it possible to determine optimal exposure modes in terms of frequency and amplitude of vibration ensuring the maximum rate of destruction.

The revealed modes are confirmed by experimental studies of ultrasonic drilling in terrestrial conditions at room temperature and when drilling in a cryogenic chamber at a temperature of -80 degrees of Celsius.

The experimental results showed that the greatest energy efficiency of drilling is achieved with the additional presence of a low-frequency impact with an attached mass and pseudo-rotational motion.

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ON THE EXTENSION OF RESERVOIR COMPUTING WITH AN INERTIAL FORM

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1. Introduction and formulation

We propose a new framework for reservoir computing [1]. In this talk, we present the essence of our proposed method and some results of the numerical experiments. We begin to formulate the reservoir with the following continuous dynamical system.

$$\begin{cases} \dfrac{\partial}{\partial t}\vec{u}(x,t) - \chi\triangle\vec{u}(x,t) - \phi\Big(\displaystyle\int_\Omega \breve{W}(x,y)\vec{u}(y,t)\,\mathrm{d}y + v(x)\vec{s}(t)\Big) = 0 \\ \qquad\qquad \text{in } \Omega_T \equiv (0,1)\times(0,T), \\ \vec{u}(x,t) = \vec{0} \quad \text{on } \partial\Omega\times(0,T), \\ \vec{u}(x,0) = \vec{u}_0(x) \text{ on } \Omega, \end{cases} \tag{1}$$

where $\vec{u}(x,t) \in \mathbb{R}^K$ denotes a vector-valued function defined on $\Omega\times\mathbb{R}_+ \equiv \Omega\times(0,\infty)$, $\triangle\vec{u} = \frac{\mathrm{d}^2\vec{u}}{\mathrm{d}x^2}$, χ is a positive constant, $v(x)$ is a function that depends only on x, $\vec{s}(t) \in \mathbb{R}^K$ is a vector-valued function that depends only on t, and $\breve{W}(x,y)$ denotes a kernel operator defined on $\Omega\times\Omega$ with some notable properties described below. We also use the notation $\|\cdot\|$ to denote an operator norm of a certain operator, and $H = \big(L_2(\Omega\times\mathbb{R}_+)\big)^K$, the norm and inner product of which are denoted by $|\cdot|$ and $(\cdot,\cdot)$, respectively, and

$$\breve{W} * \vec{u} \equiv \int_\Omega \breve{W}(x,y)\vec{u}(y,t)\,\mathrm{d}y, \quad \widetilde{A}(t)\vec{u} \equiv A_0\vec{u} - \breve{F}(\vec{u},\vec{s}).$$

$$\breve{F}(\vec{u},\vec{s}) \equiv \vec{u} + \phi\Big(\breve{W} * \vec{u} + v(x)\vec{s}(t)\Big),\ A_0\vec{u} = -\chi\triangle\vec{u} + \vec{u}.$$

Let $\{\vec{w}_j\}_{j=1}^{\infty}$ be the eigenfunctions of A_0 with a vanishing Dirichlet boundary condition that correspond to the eigenvalues in ascending order. Next, note that the inertial manifold of Eq. (1.1) is denoted by a graph of a map $\Phi : PH \to (I-P)H$, where P is a projection onto the linear subspace spanned by $\{w_j\}$. Hereinafter, we assume the following conditions, in which M_0, M_1, M_s, M_{s1}, and M_v denote positive constants.

1. $\phi \in C^1(\mathbb{R})$, $\phi' \in BUC(\mathbb{R})$, and $|\phi'(x)| \le M_1 \ \forall x \in \mathbb{R}$.
2. $f \mapsto \int_{\Omega\times\Omega} \breve{W}(x,y)f(y)\ \mathrm{d}y$ forms a bounded continuous operator from $L_2(\Omega)$, with a norm on $L_2(\Omega)$ denoted by $\|\breve{W}\|$, satisfying $\|\breve{W}\| < \chi\pi^2/M_1$.
3. $\vec{s} \in C^1(\mathbb{R}_+) \bigcap BUC(\mathbb{R}_+)$, $\nabla\vec{s} \in BUC(\mathbb{R}_+)$, and $|\vec{s}|_\infty \le M_s$ and $|\nabla\vec{s}|_\infty \le M_{s1}$.
4. $v \in BUC(\Omega)$ and $|v|_\infty \le M_v$, $|\vec{p}_0| = |P\vec{u}_0| < M_0$,
5. $\phi'(x) > 0$ on $\mathbb{R}$ and $\phi'(x) > c_1$ on $|x| < \|\breve{W}\|\sqrt{M_0^2 + (1/\chi)^2} + M_s M_v$ with some $c_1 > 0$.
6. For any family of vector-valued functions $\{\vec{g}_\sigma\}_{\sigma\in[0,1]}$ with $\vec{g}_\sigma = (g_\sigma^{(k)})$ that satisfies $\vec{g}_\sigma \in D(A)$ and $|g_\sigma^{(k)}(x,t)| < \|\breve{W}\|\sqrt{M_0^2 + (1/\chi)^2} + M_s M_v$ for $x \in \Omega$, $t > 0$, and $\sigma \in [0,1]$, $v(x)$ satisfies
$$(v, \vec{w}_l) + \sum_{j=M'+1}^{M'+N'} \lambda_j^{-1}\Big(v\Big(\int_0^1 \phi'(\vec{g}_\sigma)\ \mathrm{d}\sigma\Big) \odot \vec{w}_j\Big)\big(\breve{W} * \vec{w}_j, \vec{w}_l\big) > 0\ (l = 1, 2, \ldots, M').$$

We employ the concept of the *approximate inertial form* [2] to derive the expression obtained below.

$$\begin{cases} \dfrac{\mathrm{d}\vec{p}}{\mathrm{d}t} + A_0\vec{p} = P\breve{F}\big(\vec{p} + \Phi_0(\vec{p}, \vec{s}(t)), \vec{s}(t)\big), \\ \vec{p}\big|_{t=0} = \vec{p}_0. \end{cases} \tag{2}$$

Here, Φ_0 denotes the approximate inertial manifold [2], the definition of which will be clarified during the talk.

2. Results

In our previous paper [1], we have proven the following:

Theorem 1 *Let $T > 0$ be arbitrary, and suppose that conditions* (A)–(F) *hold. Moreover, we assume that $M_1\|\breve{W}\| + \big(1 + M_1\|\breve{W}\|\big)^2 A_0(\chi) < \chi\pi^2$, where $A_0(\chi) = \frac{\chi\left(e^{1/4\sqrt{\chi}} - e^{-1/4\sqrt{\chi}}\right)^2}{\left(e^{1/2\sqrt{\chi}} + e^{-1/2\sqrt{\chi}}\right)}$. Thus, we have the following properties.*

(i) For two initial data $\vec{p}_{0j} \in D(A_0)$ $(j = 1, 2)$, there exists a certain $\omega_1 > 0$ such that the corresponding solutions $\vec{p}^{(j)}$ $(j = 1, 2)$ of Eq. (1.2) satisfy the echo state property $\big|(\vec{p}^{(1)} - \vec{p}^{(2)})(t)\big| \le c\big|\vec{p}_{01} - \vec{p}_{02}\big|e^{-\omega_1 t} \quad t \in (0, T)$.

(ii) For two input signals $\vec{s}_j(t)$ $(j = 1, 2)$, the corresponding solutions $\vec{p}_j$ $(j = 1, 2)$ of Eq. (1.2) satisfy

$$\int_0^\infty \big|(\vec{p}_1 - \vec{p}_2)(t)\big|^2\ \mathrm{d}t \ge c(\chi)\sum_{l=1}^{M'}\int_0^\infty \Big|\int_0^t e^{-\lambda_l(t-t')}\vec{V}_2(t')^\top \tilde{s}(t')\mathrm{d}t'\Big|^2\ \mathrm{d}t,$$

where $\tilde{s} = \vec{s}_1(t) - \vec{s}_2(t)$, $\vec{V}_2(t)$ *denotes a vector-valued function whose components are all positive for* $t > 0$, *and* $c(\chi)$ *denotes a positive constant that depends on* χ.

We have implemented the discretized version of this scheme and an output layer that contains a linear unit. Some types of regularization (ridge, lasso, and elastic net) can also be implemented in this context. In this talk, we present some results of the numerical experiments conducted using these output layers with some actual datasets.

REFERENCES

1. Honda, H., "Reservoir Computing with an Inertial Form," SIAM J. Appl. Dyn. Syst., 20(2021), 1320–1347.
2. Jolly, M. S., Kevrekidis, I. G. and Titi, E. S., Preserving dissipation in approximate inertial forms for the Kuramoto–Sivashinsky equation, J. Dyn. Diff. Equat. **2** (1991), 179–197.

SEMIGROUP AND MAXIMAL REGULARITY APPROACH TO THE PRIMITIVE EQUATIONS

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We review our results so far obtained for the primitive equations (PEs), which describe large-scale motion of ocean or atmosphere. In the simplified form, they read:

$$\begin{aligned} \partial_t v - \Delta v + (u\cdot\nabla)v + \nabla_H p &= 0 \quad \text{on } \Omega\times(0,\infty),\\ \partial_z p &= 0 \quad \text{on } \Omega\times(0,\infty),\\ w|_{z=-h,0} &= 0 \quad \text{on } \mathbb{T}^2,\\ v(0) &= a \quad \text{on } \Omega,\\ \operatorname{div} u &= 0 \quad \text{on } \Omega\times(0,\infty), \end{aligned}$$

where $\Omega = \mathbb{T}^2\times(-h,0)$, $u = (v,w) = (v_1,v_2,w)$ is the 3D velocity, p means the pressure, and $\nabla_H = (\partial_x,\partial_y)^\top$. Compared with the 3D incompressible Navier–Stokes equation, one can see that there is no evolution law described for the vertical velocity w and that p is independent of z, implying the hydrostatic balance.

Our results are concerned with the global-in-time existence and uniqueness of a strong solution provided by an analytic semigroup approach or by a maximal-regularity theory. In particular, investigation of the linear part of the PEs, i.e., the *hydrostatic Stokes operator*, has a central importance. We will first present the L^p-theory where the strong solution is constructed for initial data belonging to $H^{2/p,p}$. We show that the solution becomes C^∞ (even real analytic) in x and t after initial time. Then the endpoint case $p=\infty$ (more precisely, an anisotropic space $L^\infty_{xy}L^p_z$ will be considered) is discussed, which requires more delicate arguments due to the lack of boundedness in L^∞ of the hydrostatic Helmholtz projector. Finally, justification of hydrostatic approximation in the L^p-setting, that is, convergence from the Navier–Stokes equations to the PEs in the zero aspect-ratio limit, will also be mentioned.

This talk is based on a series of papers collaborated with Prof. Hieber, Prof. Hussein, Prof. Giga, Dr. Gries, Dr. Wrona, and Dr. Furukawa.

REFERENCES

1. M. Hieber and T. Kashiwabara, *Global strong well-posedness of the three dimensional primitive equations in L^p-spaces*, Arch. Rational Mech. Anal. **221** (2016), 1077–1115.
2. Y. Giga, M. Gries, M. Hieber, A. Hussein and T. Kashiwabara: *The hydrostatic Stokes semigroup and well-posedness of the primitive equations on spaces of bounded functions*, J. Funct. Anal. **279** (2020), doi: 10.1016/j.jfa.2020.108561
3. K. Furukawa, Y. Giga, M. Hieber, A. Hussein, T. Kashiwabara, and M. Wrona: *Rigorous justification of the hydrostatic approximation for the primitive equations by scaled Navier–Stokes equations*, Nonlinearity **33** (2020), 6502–6516.

ON POLYNOMIAL COMPACTNESS OF ELASTIC NEUMANN–POINCARÉ OPERATORS ON $C^{1,\alpha}$ BOUNDARIES IN THREE DIMENSIONS

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1. Introduction

The elastic Neumann–Poincaré (eNP) operator is a boundary integral operator that appears naturally when we solve classical boundary value problems for the Lamé system using layer potentials. Recently, there is rapidly growing interest in the spectral properties of the eNP operator in relation to cloaking by anomalous localized resonance (CALR) [3]. Anomalous localized resonance occurs at the accumulation point of eigenvalues, which motivates us to investigate the spectral structure of the eNP operator.

The Lamé system, a system of equations of linear elasticity, is described by

$$\mathcal{L}_{\lambda,\mu}u := \mu\Delta u + (\lambda+\mu)\nabla\nabla\cdot u = f,$$

where $u = (u_1, \dots, u_d)$ $(d = 2, 3)$ is the displacement, (λ, μ) are the Lamé constants, and f is the force term. In what follows, we assume that the pair of constants (λ, μ) satisfies the strong convexity condition:

$$\mu > 0, \quad d\lambda + 2\mu > 0.$$

Let $\mathbf{\Gamma}(x) = (\Gamma(x))_{i,j=1}^d$ be the fundamental solution to the Lamé system associated with the Lamé constants (λ, μ), namely,

$$\Gamma_{ij}(x) := \begin{cases} \dfrac{\alpha_1}{2\pi}\delta_{ij}\log|x| - \dfrac{\alpha_2}{2\pi}\dfrac{x_i x_j}{|x|^2}, & d = 2, \\ -\dfrac{\alpha_1}{4\pi}\dfrac{\delta_{ij}}{|x|} - \dfrac{\alpha_2}{4\pi}\dfrac{x_i x_j}{|x|^3}, & d = 3, \end{cases} \qquad |x| \neq 0,$$

where

$$\alpha_1 := \frac{1}{2}\left(\frac{1}{\mu} + \frac{1}{\lambda+2\mu}\right), \quad \alpha_2 := \frac{1}{2}\left(\frac{1}{\mu} - \frac{1}{\lambda+2\mu}\right).$$

Let Ω be a bounded domain in $\mathbb{R}^d$ with a Lipschitz boundary $\partial\Omega$. For a vector-valued function u, the conormal derivative $\partial_\nu u$ corresponding to the Lamé system is defined by

$$\partial_\nu u := \lambda(\nabla \cdot u)n + 2\mu(\widehat{\nabla} u)n,$$

where n is the outward unit normal to $\partial\Omega$ and $\widehat{\nabla} u$ is the symmetric gradient of the vector-valued function u, namely

$$\widehat{\nabla} u := \frac{1}{2}\left(\nabla u + (\nabla u)^T\right).$$

Here, $(\nabla u)^T$ denotes the transpose of the matrix ∇u. Then, the eNP operator $\mathbf{K}^*$ is defined by

$$\mathbf{K}^*[f](x) := \text{p.v.} \int_{\partial\Omega} \partial_{\nu_x} \mathbf{\Gamma}(x-y) f(y)\, d\sigma_y, \qquad \text{a.e. } x \in \partial\Omega.$$

Here, we consider the conormal derivative $\partial_{\nu_x}\mathbf{\Gamma}(x-y)$ of the matrix columnwise and p.v. stands for the Cauchy principal value.

2. Main Results

The main result in this talk is the following.

Theorem 2 *Let Ω be a bounded domain in $\mathbb{R}^3$ with the $C^{1,\alpha}$-smooth boundary for some $\alpha > 0$. Let $\mathbf{K}^*$ be the eNP operator on $\partial\Omega$ corresponding to the pair of Lamé constants (λ, μ). Let $p_3(t) := t(t+\kappa_0)(t-\kappa_0)$, where κ_0 is given by*

$$\kappa_0 := \frac{\mu}{2(\lambda + 2\mu)}.$$

Then, $p_3(\mathbf{K}^*)$ is compact on $H^{-1/2}(\partial\Omega)^3$. Moreover, $\mathbf{K}^*(\mathbf{K}^* + \kappa_0 I)$, $\mathbf{K}^*(\mathbf{K}^* - \kappa_0 I)$ and $(\mathbf{K}^*)^2 - \kappa_0^2 I$ are not compact on $H^{-1/2}(\partial\Omega)^3$.

From Theorem 2 and the spectral mapping theorem, we obtain the following result on the asymptotic behavior of eigenvalues.

Corollary 1 *The spectrum of $\mathbf{K}^*$ on $H^{-1/2}(\partial\Omega)^3$ consists of three non-empty sequences of eigenvalues which converge to 0, κ_0 and $-\kappa_0$, respectively.*

Theorem 2 was once proved in [2] by assuming C^∞-smoothness on the boundary $\partial\Omega$. On the other hand, in the two dimensional case, Ando et al [1] proved the following proposition, which motivated us to prove Theorem 2 on $C^{1,\alpha}$ boundaries.

Proposition 1 *Let Ω be a bounded domain in $\mathbb{R}^2$ with the $C^{1,\alpha}$-smooth boundary for some $\alpha > 0$. Let $\mathbf{K}^*$ be the eNP operator on $\partial\Omega$ corresponding to the pair of Lamé constants (λ, μ). Let $p_2(t) := (t+\kappa_0)(t-\kappa_0)$. Then, $p_2(\mathbf{K}^*)$ is compact on $H^{-1/2}(\partial\Omega)^3$.*

The key idea to the proof of Theorem 2 is to "approximate" the eNP operator by the surface Riesz transforms R_j^g, which are defined by

$$R_j^g[f](u) = \frac{1}{2\pi}\text{p.v.} \int_{\mathbb{R}^2} (u_j - v_j)\langle u-v, G(u)(u-v)\rangle^{-3/2} f(v) dv, \quad j = 1, 2.$$

Here, $G(u) = (g_{ij}(u))_{i,j=1,2}$ is a positive-definite symmetric matrix valued function on $\mathbb{R}^2$ and the bracket $\langle \cdot, \cdot \rangle$ implies the inner product on $\mathbb{R}^2$.

This result was obtained by a joint work with Hyeonbae Kang (Inha University), and it was supported by NRF grants No. 2016R1A2B4011304 and 2017R1A4A1014735.

REFERENCES

1. Ando K., Ji Y.-G., Kang H., Kim K., and Yu S. *Spectral properties of the Neumann–Poincaré operator and cloaking by anomalous localized resonance for the elasto-static system*, Eur. J. App. Math., **29**, 2018, pp. 189–225.
2. Ando K., Kang H., and Miyanishi Y. *Elastic Neumann–Poincaré Operators on three dimensional smooth domains: polynomial compactness and spectral structure*, Int. Math. Res. Not., **2019**, 2019, no. 12, pp. 3883–3900.
3. Li H. *Recent progress on the mathematical study of anomalous localized resonance in elasticity*, Elec. Res. Arch., **28**, 2020, no. 3, pp. 1257–1272.

INVERSE PROBLEMS FOR ELASTIC BODY WITH CLOSELY LOCATED THIN INCLUSIONS

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We consider an equilibrium problem for a 2D elastic body with two thin closely located elastic inclusions. Inclusions are in a contact with each other which means a presence of a crack between them. Nonlinear boundary conditions of inequality type are imposed at the crack faces providing a mutual non-penetration. Moreover, the inclusions cross the external boundary of the elastic body. The unique solvability of the problem is proved. Passages to limits are investigated as rigidity parameters of the inclusions tend to infinity, and limit models are analyzed. Inverse problems for finding the rigidity parameter and Lamé parameters of the elastic body are investigated with a boundary measurement of the tip point displacement of the inclusion.

REFERENCES

1. Khludnev A. M. *Equilibrium of an elastic body with closely spaced thin inclusions.* Comp. Math. and Math. Physics. 2018. V. 58. N 10. P. 1660–1672.
2. Khludnev A. M. *Inverse problems for elastic body with closely located thin inclusions.* Z. Angew. Math. Phys. 2019. 70:134.

MINIMIZING MOVEMENTS FOR MEAN CURVATURE FLOW OF PARTITIONS

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In this talk I will discuss a weak notion of mean curvature flow of networks and space partitions obtained via the minimizing movements (Almgren-Taylor-Wang-De Giorgi) method, approximating the evolution with a sequence of discrete minimum problems associated to the perimeter perturbed by a non-symmetric bulk term. Besides the existence, I discuss some properties of the flow, such as time-continuity, comparison properties, and also consistency with classical solutions in some special cases. Also I will address some qualitative properties of the flow for long times.

GEOMETRICALLY EXACT INTEGRAL-BASED NONLOCAL MODEL OF DUCTILE DAMAGE AND FRACTURE: BASIC PROPERTIES AND NUMERICAL TREATMENT

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We develop the integral-based approach to nonlocal damage and fracture in ductile materials [2]. The starting point is the phenomenological model of finite strain plasticity with ductile damage [3]. This model is delocalized by applying the integral-based averaging operator. It smoothens the damage-related parameters like continuity and porosity. At least one internal length-scale parameter in introduced into the formulation, depending on the implemented averaging procedure. Thus, the damage localization is constrained by the presence of length-like parameters. This constraint regularizes the initial boundary value problem. The regularization enables numerically robust and physically plausible simulations of crack initiation and propagation [1]. Spurious localization of damage and plastic strain into a zero thickness band is prevented. The basic properties of the new non-local material model are analysed theoretically: We show that the model is thermodynamically consistent, objective, and w-invariant. Robust numerical algorithms are suggested. The model is implemented into an academic non-linear finite element code. As a demonstration problem, we simulate crack initiation and propagation in a plate with a hole. The simulation results agree with actual experiment in terms of applied force, plastic zone evolution, and cracking patterns. Another series of tests includes fracture of compact tension specimen. The force-displacement curves, estimated fracture toughness, and damage patterns showcase the mechanical phenomena, captured by the modelling framework. The impact of the constitutive assumptions on the predicted structural strength is studied: Isotropic and anisotropic delocalization is discussed; the averaging is applied on current and reference configurations; continuity is employed as a dual damage variable to restrain the nonphysical diffusion of damage.

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REFERENCES

1. Klyuchancev V, S. and Shutov A. V. *Nonlocal FEM Simulations of ductile damage with regularized crack path predictions.* J. Phys.: Conf. Ser. 2021. V. 1945. 012018.
2. Shutov A. V., Klyuchantsev, V. S. *Large strain integral-based nonlocal simulation of ductile damage with application to mode-I fracture.* International Journal of Plasticity. V. 144. 103061.
3. Shutov A. V., Silbermann C. B., Ihlemann J. *Ductile damage model for metal forming simulations including refined description of void nucleation.* International Journal of Plasticity. V. 71. P. 195-217.

POROELASTIC MEDIUM WITH NON-PENETRATING CRACK DRIVEN BY HYDRAULIC FRACTURE

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A new class of unilateral variational models appearing in the theory of poroelasticity is introduced and studied. A poroelastic medium consists of solid phase and pores saturated with a Newtonian fluid. The medium contains a fluid-driven crack, which is subjected to non-penetration between the opposite crack faces. The fully coupled poroelastic system includes elliptic-parabolic governing equations under the unilateral constraint. Well-posedness of the corresponding variational inequality is established based on the Rothe semi-discretization in time, after subsequent passing time step to zero. The NLCP-formulation of non-penetration conditions is given which is useful for a semi-smooth Newton solution strategy.

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REFERENCES

1. Khludnev A. M. and Kovtunenko V. A. *Analysis of Cracks in Solids*. WIT-Press, Southampton, Boston 2000.
2. Kovtunenko V. A. Poroelastic medium with non-penetrating crack driven by hydraulic fracture: variational inequality and its semidiscretization// *J. Comput. Appl. Math.*. 2021. accepted.

OPTIMAL CONTROL OF THE LOCATION OF THE HINGE POINT OF RIGID INCLUSIONS IN AN EQUILIBRIUM PROBLEM OF A TIMOSHENKO PLATE

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We consider a family of contact problems on the equilibrium of a Timoshenko composite plate containing two thin rigid inclusions, which are connected in a hinged manner. The family's problems depends on a parameter specifying the coordinate of a connection point of the inclusions. An optimal control problem is formulated with a quality functional defined using an arbitrary continuous functional given on a suitable Sobolev space. In this case, control is specified by the coordinate parameter of the connection point of the inclusions. The continuity of solutions of the family's problems on this parameter is proved. The solvability of the optimal control problem is established.

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REGULARITY AT THE CRACK-TIP FOR MUMFORD-SHAH MINIMIZERS: SURGERY VS VARIATIONS

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In this presentation I will talk about some recent results in the regularity of the discontinuity set at the crack-tip/-front for the minimizers of the Mumford-Shah functional in $\mathbf{R}^2$ and $\mathbf{R}^3$.

The main focus will be on the new Euler-Lagrange condition at the crack-tip for the minimizers of the Mumford-Shah functional in the plane, which was discovered in [1] and [2]. The original proof based on geometric "surgery" has been recently simplified in [3] due to a rather sophisticated variational approach.

If the time allows I will also touch the numerical machinery developed in [5] for the 2-D case, as well as some surprising results obtained in [4] for the crack-front in 3-D.

REFERENCES

1. John Andersson, Hayk Mikayelyan *The asymptotics of the curvature of the free discontinuity set near the cracktip for the minimizers of the Mumford-Shah functional in the plain.* arXiv:1204.5328
2. John Andersson, Hayk Mikayelyan *Regularity up to the Crack-Tip for the Mumford-Shah problem.* arXiv:1512.05094
3. Camillo De Lellis, Matteo Focardi, Silvia Ghinassi *Endpoint regularity for 2-D Mumford-Shah minimizers: On a theorem of Andersson and Mikayelyan.* Journal de Mathématiques Pures et Appliquées. Volume 155, November 2021, Pages 83-110
4. Lemenant, Antoine; Mikayelyan, Hayk *Stationarity of the crack-front for the Mumford-Shah problem in 3D.* J. Math. Anal. Appl. 462 (2018), no. 2, 1555-1569
5. Li, Zhilin; Mikayelyan, Hayk *Fine numerical analysis of the crack-tip position for a Mumford-Shah minimizer.* Interfaces and Free Bound. 18 (2016), no. 1, 1-18

VIBROSEIS REFLECTION EXPLORATION FOR ANISOTROPIC MEDIA

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Let $x = (x_1, x_2, x_3)$ be a point in a bounded domain $\Omega \subset \mathbb{R}^3$ with Lipschitz boundary Γ. We consider Ω as a reference domain for a linear elastic medium with elasticity tensor $C = (C_{ijkl}(x)) \in L^\infty(\Omega)$. It is physically natural to assmume the following symmetry and strong convexity conditions for C:
for any a.e. $x \in \overline{\Omega}$ and indices $i, j, k, l \in \{1, 2, 3\}$, it satisfies
symmetry:

$$\begin{cases} C_{ijkl}(x) = C_{ijlk}(x) \ (\textit{minor symmetry}), \\ C_{ijkl}(x) = C_{klij}(x) \ (\textit{major symmetry}) \end{cases}$$

strong convexity:

$$\begin{array}{l} {}^{\exists}\delta > 0 \text{ s.t. for any symmetric matrix } \epsilon = (\epsilon_{ij}), \\ \epsilon : (C :: \epsilon) = \sum_{i,j,k,l=1}^{3} C_{ijkl}(x)\epsilon_{ij}\epsilon_{kl} \geq \delta(\epsilon : \epsilon) \text{ (a.e. } x \in \Omega) \end{array}$$

The (linear) displacement vector $u = (u_1, u_2, u_3)$ satisfies the elasticity equations of system

$$\begin{aligned} (\rho\partial_t^2 u - L_C u)_i &= (\rho\partial_t^2 u - \operatorname{div}(C :: \nabla u))_i \\ &:= \rho\partial_t^2 u_i - \sum_{j,k,l=1}^{3} \partial_j \left(C_{ijkl}(x)_l u_k\right) = 0,\ 1 \leq i \leq 3, \text{ in } \Omega_T = \Omega \times (0, T), \end{aligned}$$

where $\partial_j = \partial/(\partial x_j)$, $0 < \rho_0 \leq \rho \in L^\infty(\Omega)$ is the density bounded from below by a constant ρ_0.

Now let $\Sigma \subset \Gamma^N$ be an non-empty, open, connected set, $\Sigma_T = \Sigma \times (0, T)$ and $\dot{H}^{-1/2}(\overline{\Sigma}) := \{g \in \overline{H}^{-1/2}(\Gamma) : \operatorname{supp} g \subset \overline{\Sigma}\}$. Consider

$$(\mathrm{MP}) \begin{cases} (\rho\partial_t^2 u - L_C)u = 0 & \text{in } \Omega_T, \\ \partial_C u := (C :: \nabla u)\nu = g \in H^1((0, T); \dot{H}^{-1/2}(\overline{\Sigma}) & \text{on } \Gamma_T, \\ u\big|_{t=0} = 0 \in H^1(\Omega),\ \partial_t u\big|_{t=0} = 0 \in L^2(\Omega), & \text{in } \Omega, \end{cases}$$

where ν is the outer unit normal of Γ.

Well-posedness of (MP) : There exists a unique solution $u \in L^\infty((0, T); H^1(\Omega))$ with $\partial_t u \in L^\infty((0, T); L^2(\Omega))$, $\partial_t^2 u \in L^\infty((0, T); (H^1(\Omega)')$ and it depends continuously on g.

Based on the well-posedness of (MP), we defined the *localized Neumann-to-Dirichlet map=ND map* as follows.

ND map $\Phi_{\rho,C}^{T,\Sigma}$:

$$\Phi_{\rho,C}^{T,\Sigma} : H^1((0, T); \dot{H}^{-1/2}(\overline{\Sigma})) \ni g \mapsto u^g\big|_{\Sigma_T} \in H^1((0, T); \overline{H}^{1/2}(\Sigma))$$

with the solution $u = u^g$ of $(\mathrm{MP})_0$.

Further assumptions on the density and elastic tensor:

- finitely many subdomains $D_\alpha \subset \Omega$, $\alpha \in A$, s.t.

$$\bar{\Omega} = \cup_{\alpha \in A}\overline{D_\alpha}, \quad D_\alpha \cap D_\beta = \emptyset \text{ if } \alpha \neq \beta$$

  $\{D_\alpha\}_{\alpha \in A}/\{\overline{D_\alpha}\}_{\alpha \in A}$ and each ∂D_α are called *cover* of Ω and *interface*, respectively. Each interface is Lipschitz smooth.

- ρ, C are homogeneous (i.e. constant) in each D_α.

  Refer these as saying (ρ, C) is "piecewise homogeneous".

Inverse problem: Show the uniqueness of identifying the density ρ and elasticity tensor C by knowing $\Phi_{\rho,C}^{T,\Sigma}$. Especially identify in a region of interest (ROI) with a time small as possible.

This is a typical question asked for the vibroseis exploration technique in reflection seismology.

Our aim is to give some result which give some mathematical foundation for the vibroseis exploration technique.

Main results: There exists a time T which is possible to estimate and gives affirmative answers to the aforementioned inverse problem as follows.

- If the interfaces are known, we have the uniqueness under "curvature condtion" on the interfaces (i.e. it is locally C^1 and the image of Gauss map contains the image of a non-constant continuous curve).
- If the interfaces are unknown, we have the uniqueness under "strong curvature condition" (i.e the curvature condition holds everywhere on each interface which doesn't touch $\partial\Omega$) and piecewise analytic condition on the interfaces. The curvature condition and strong curvature condition can be removed if C is transversally isotropic (i.e. the case C has only one symmetric axis).

Remark
(i) If ρ, C are analytic on each $\overline{D_\alpha}$ and C is isotropic, the curvature condition and strong curvature condition can be removed in the above statements of the main results.
(ii) The keys to proving the main results and the above (i) are the propagation of the ND-map and the theory of subanalytic sets.

REVISITING J-INTEGRAL IN FRACTURE MECHANICS

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We start by revisiting J-integral in Rice[12, 13] from a mathematical point of view. Consider a linear or nonlinear elastic body (u, ε, σ) under body force f, surface force g and subjected to 2D deformation; $u = (u_i(x_1, x_2))_{i=1,2}, \varepsilon_{ij}(u) = (\partial u_i/\partial x_j + \partial u_j/\partial x_i)/2, \sigma_{ij} = \partial\widehat{W}(x, \varepsilon)/\varepsilon_{ij}, i, j = 1, 2$. Here $\widehat{W}(x, \varepsilon)$ is strain-energy density. Considering that there is a defect Σ inside a smooth domain Ω, the constitutive equation of the elastic body is defined on $\Omega_\Sigma := \Omega \setminus \Sigma$, fixed on $\Gamma_D \subset \partial\Omega$, g is given on $\Gamma_N := \partial\Omega \setminus \Gamma_D$, and stress free on Σ. Here, consider a crack or a notch as defects Σ.

For a vector field μ, by the *chain rule* and *symmetry of second derivatives*, we have

$$\mu \cdot \nabla_x \widehat{W}(x, \varepsilon) = \mu \cdot \nabla_\xi \widehat{W}(\xi, \varepsilon)|_{\xi=x} + \sigma : \varepsilon(\mu \cdot \nabla u) - \sigma : (\nabla\mu\nabla u) \quad (1)$$

By *Green's formula*, we obtain for a domain $A \cap \Omega_\Sigma$

$$\int_{A\cap\Omega_\Sigma} \sigma : \varepsilon(\mu \cdot \nabla u)dx = \int_{A\cap\Omega_\Sigma} f \cdot (\mu \cdot \nabla u) + \int_{\partial(A\cap\Omega_\Sigma)} T \cdot (\mu \cdot \nabla u)ds \quad (2)$$

$$\int_{A\cap\Omega_\Sigma} \mu \cdot \nabla_x \widehat{W}(x, \varepsilon)dx = \int_{\partial(A\cap\Omega_\Sigma)} \widehat{W}(x, \varepsilon)\mu \cdot n\, ds - \int_{A\cap\Omega_\Sigma} \widehat{W}(x, \varepsilon)\mathrm{div}\mu\, dx \quad (3)$$

where n is the outward unit normal of $\partial(A\cap\Omega_\Sigma)$ and $T=\sigma n$ the surface force. *J-integral* J_C and its path independence are derived in [12] using the virtual work principle and the divergence theorem when $\mu=e_1:=(1,0)$ where elasticity is homogeneous $\widehat{W}(x,\varepsilon)=\widehat{W}(\varepsilon)$, $f=0$, In μ, (2) and (3) are *the principle of virtual work* and *the divergence theorem*, repectively.

We now define *Generalize J-integal (GJ-integral)* $J_A(u,\mu):=P_A(u,\mu)+R_A(u,\mu)$ by

$$P_A(u,\mu) := \int_{\partial(A\cap\Omega_\Sigma)}\{\widehat{W}(x,\varepsilon)\mu\cdot n - T\cdot(\mu\cdot\nabla u)\}ds \tag{4}$$

$$\begin{aligned} R_A(u,\mu) &:= -\int_{A\cap\Omega_\Sigma}\{\mu\cdot\nabla_\xi\widehat{W}(x,\varepsilon)+f\cdot(\mu\cdot\nabla u)\}dx \\ &\quad+\int_{A\cap\Omega_\Sigma}\{\sigma:(\nabla\mu\nabla u)-\widehat{W}(x,\varepsilon)\mathrm{div}\mu\}dx \end{aligned} \tag{5}$$

Denoting by $A(C)$ the domain surrounded by the closed path C, $J_C=P_{A(C)}(u,e_1)$ if elasticity is homogeneous and $f=0$. The following theorem corresponds to the path independence of the J-integral in [12].

Theorem 3 *If (1), the principle of virtual work (2) and the divergence theorem (3) hold, then*

$$J_A(u,\mu)=0 \qquad \textit{for all } \mu\in W^{1,\infty}(\mathbb{R}^2;\mathbb{R}^2) \tag{6}$$

GJ-integral is proposed, and Theorem 3 is proved in [9]. In [12], the following was shown which suggested that the energy release rate $\mathcal{G}=J_C$ from [3]. Cherepanov [1] also derive J-integral and $\mathcal{G}=J_C$ by a consideration of the *law of conservation of energy* during crack growth.

Theorem 4 *Assume that Σ is the crack along x_1-axis. For an isotropic elastic body, if the closed path C surrounds the crack tip of Σ*

$$J_{A(C)}(u,e_1)=J_C=\frac{1}{E'}(K_I^2+K_{II}^2) \tag{7}$$

where $E'=E$(plane stress)$;=E/(1-\nu^2)$(plane strain), K_1,K_{II} are stress intensity factors (SIFs) for the opening, in-plane sliding,

By Theorem 4 and the contraposition of Theorem 3, *(1)-(3) does not hold* when Σ is a crack, and $\int_{A(C)}e_1\cdot\nabla\widehat{W}(\varepsilon)dx=\infty$ if SIF is non-zero. Therefore, it is difficult to prove $\mathcal{G}=J_C$ for a crack using mechanics. In [13], $\mathcal{G}=J_C$ is proved when Σ is smooth notch. The first mathematical proof seems to be $\mathcal{G}=R_{\Omega_\Sigma}(u,\beta e_1)=J_C$ in [2] where β is the cut-off function of the crack tip. In the same year, it is proved that $\mathcal{G}=J_A(u,\mu_C)\left(\int_{\partial\Sigma}\dot{\Sigma}d\lambda\right)^{-1}$ when Σ is 2-dimensional smooth manifold in $\mathbb{R}^3$ with the edge $\partial\Sigma$ and grow smoothly (μ_C the vector field obtained from growth and $\dot{\Sigma}$ speed) in [9]. Considering Hadamard's variational formula, it is proved that $-d\mathcal{E}/dt=R_{\Omega_\Sigma}(u,\mu_\phi)$ in [10], when the singulr points moving $\gamma\mapsto\phi(\gamma)$ inside Ω in linear problems. Later using the abstract theory in Banach space by Kimura[7], $-d\mathcal{E}/dt=R_{\Omega_\Sigma}(u,\mu_\phi)$ is proved in [11] in nonlinear case. As far as I know, there are some in the references as mathematical studies on $\mathcal{G}=R_{\Omega_\Sigma}(u,\beta e_1)=J_C$ under non-penetration conditions or nonlinear elasticity. In addition, the wrong use of J-integral in engineering and the numerical calculation by FEM will be talked.

REFERENCES

1. Cherepanov, G.P., On crack propagation in continuous media, Prikl. Math. Mekh., 31 (1967), 476–493.
2. Destuynder, Ph. and Djaoua,M., Sur une interprétation de l'intégrale de Rice en théorie de la rupture fragile. Math. Meth. in Appl. Sci., 3 (1981), 70–87.
3. Irwin, C.R., Fracture, in Handbuch der Physik, Vol. VI, Springer, Berlin, 1958.
4. Knees, D., Griffith-formula and J-integral for a crack in a power-law hardening material. Math. Models Meth. Appl. Sci., 16(2006), 1723–1749.
5. Knees, D. and Mielke, A., Energy release rate for cracks in finite-strain elasticity, Math. Meth. Appl. Sci. 2008(31), 501–528.
6. Khludnev, A. M., and Sokolowski, J. The Griffith formula and the Rice- Cherepanov integral for crack problems with unilateral conditions in nonsmooth domains. Eur. J. Appl. Math. 10(1999), 379–394.
7. Kimura, M.: Shape derivative of minimum potential energy: abstract theory and applications, Jindvrich Nevcas Center for Mathematical Modeling Lecture notes, Vol. 4, 1–38, matfyzpress, 2008
8. Kovtunenko, V,A,, Primal-dual methods of shape sensitivity analysis for curvilinear cracks with nonpenetration, IMA J. Appl. Math. 71 (2006), 635–657.
9. Ohtsuka, K., Generalized J-integral and three dimensional fracture mechanics I, Hiroshima Math. J., 11(1981), 21–52.
10. Ohtsuka,K., Generalized J-integral and its applications. I. – Basic theory, Japan J. Appl. Math., 2 (1985), 329–350.
11. Ohtsuka. K. and Kimura, M., Differentiability of potential energies with a parameter and shape sensitivity analysis for nonlinear case: the p-Poisson problem, Japan J. Indust. Appl. Math., 29 (2012), 23–35.
12. Rice, J.R., A path-independent integral and the approximate analysis of strain concentration by notches and cracks, J. Appl. Mech., 35 (1968), 379–386.
13. Rice, J.R., Mathematical analysis in the mechanics of fracture, Fracture Volume II, Academic Press, 1968, 191–311.

IMPROVED APPROXIMATIONS IN HOMOGENIZATION OF HIGHER ORDER ELLIPTIC OPERATORS

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In the space $\mathbb{R}^d$, $d\geq 2$, we consider divergence-type operators of an even order $2m\geq 4$

$$A_\varepsilon = (-1)^m \sum_{|\alpha|=|\beta|=m} D^\alpha(a^\varepsilon_{\alpha\beta}(x)D^\beta) \tag{1}$$

with ε-periodic coefficients $a^\varepsilon_{\alpha\beta}(x) = a_{\alpha\beta}(y)|_{y=\varepsilon^{-1}x}$, $\varepsilon > 0$ is a small parameter. Here, $\alpha=(\alpha_1,\dots,\alpha_d)$ is a multiindex of length $|\alpha|=\alpha_1+\ldots+\alpha_d$, $\alpha_j\in\mathbb{Z}_{\geq 0}$, $D^\alpha = D_1^{\alpha_1}\dots D_d^{\alpha_d}$, $D_i = D_{x_i}$, $i = 1,\dots,d$; the coefficients $a_{\alpha\beta}(y)$ are real, measurable, 1-periodic with periodicity cell $Y=[-1/2,1/2)^d$, and satisfy the conditions

$$\begin{gathered} \|a_{\alpha\beta}\|_{L^\infty(Y)} \leq \lambda_1, \quad \forall \alpha,\, \beta,\, |\alpha| = |\beta| = m, \\ \int_{\mathbb{R}^d} \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x) D^\beta\varphi D^\alpha\varphi\, dx \geq \lambda_0 \int_{\mathbb{R}^d} \sum_{|\alpha|=m} |D^\alpha\varphi|^2\, dx \quad \forall \varphi \in C_0^\infty(\mathbb{R}^d) \end{gathered} \tag{2}$$

for some positive constants λ_0 and λ_1. Operators of the type (1) with $m=2$ appear in the study of elastic thin plates made of composite materials with periodic structure.

With the family A_ε we associate the homogenized operator $\hat{A}$ of the same class (2)

$$\hat{A} = (-1)^m \sum_{|\alpha|=|\beta|=m} D^\alpha \hat{a}_{\alpha\beta} D^\beta, \tag{3}$$

where the coefficients $\hat{a}_{\alpha\beta}$ are constant and can be expressed in terms of the solutions to the problems on the periodicity cell Y. Over the last decade, there has been a great interest in approximation of the resolvent $(A_\varepsilon+1)^{-1}$ in different operator norms and estimation of errors with respect to the parameter ε. In particular, as is proved in [1],[2],

$$\|(A_\varepsilon + 1)^{-1} - (\hat{A} + 1)^{-1}\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \le C\varepsilon, \tag{4}$$

$$\|(A_\varepsilon + 1)^{-1} - (\hat{A} + 1)^{-1} - \varepsilon^m \mathcal{K}_\varepsilon\|_{L^2(\mathbb{R}^d)\to H^m(\mathbb{R}^d)} \le C\varepsilon, \tag{5}$$

where the constant C depends only on the dimension d and constants λ_0 and λ_1 in (2). The correcting operator $\mathcal{K}_\varepsilon$ is obtained by using the solutions to the problems on the periodicity cell which are introduced to compute coefficients of the operator $\hat{A}$. Note that

$$\|\varepsilon^m \mathcal{K}_\varepsilon\|_{L^2(\mathbb{R}^d)\to H^m(\mathbb{R}^d)} \le c, \quad \|\mathcal{K}_\varepsilon\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \le c. \tag{6}$$

By $(6)_2$, the estimate (4) in the operator $L^2(\mathbb{R}^d)$-norm can be obtained by simplifying (5). Due to a more delicate application of (5) in [3],[4], the L^2-estimate (4) was improved in the case of symmetric coefficients $a_{\alpha\beta}(y)$, namely,

$$\|(A_\varepsilon + 1)^{-1} - (\hat{A} + 1)^{-1}\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \le C\varepsilon^2; \tag{7}$$

without the symmetry condition on coefficients, the ε^2-order approximation of the resolvent $(A_\varepsilon + 1)^{-1}$ in the operator L^2-norm was found in the following form: $(A_\varepsilon + 1)^{-1} = (\hat{A} + 1)^{-1} + \varepsilon\mathcal{K}_1 + O(\varepsilon^2)$, where the correcting operator $\mathcal{K}_1$ is independent of ε (unlike its counterpart in (5)), but is also defined in terms of solutions to the above-mentioned main problems on the cell.

Now we are interested in ε^2-order approximations of the resolvent $(A_\varepsilon + 1)^{-1}$ in the energy operator norm, i.e., $(L^2 \to H^m)$-norm. To this end, we introduce the following homogenized operator, slightly different from that in (3):

$$\hat{A}_\varepsilon = (-1)^m \sum_{|\alpha|=m} D^\alpha \Big(\sum_{|\beta|=m} \hat{a}_{\alpha\beta} D^\beta + \varepsilon \sum_{|\delta|=m+1} b_{\alpha\delta} D^\delta \Big).$$

Here, the coefficients $\hat{a}_{\alpha\beta}$ are the same as in (3) and $b_{\alpha\delta}$ are constants defined via solutions to the additional cell problems. It is clear that $\hat{A}_\varepsilon = \hat{A} + \varepsilon B$, where the differential operator B has order $2m + 1$. One can show that the resolvent $(\hat{A}_\varepsilon + 1)^{-1}$ is a bounded operator from $L^2(\mathbb{R}^d)$ to $H^{2m}(\mathbb{R}^d)$. We define the correcting operators

$$\mathcal{K}_m(\varepsilon) = \sum_{|\gamma|=m} N_\gamma\Big(\frac{x}{\varepsilon}\Big) D^\gamma S^\varepsilon (\hat{A}_\varepsilon + 1)^{-1}, \quad \mathcal{K}_{m+1}(\varepsilon) = \sum_{|\delta|=m+1} N_\delta\Big(\frac{x}{\varepsilon}\Big) D^\delta S^\varepsilon S^\varepsilon (\hat{A}_\varepsilon + 1)^{-1},$$

where we use the Steklov smoothing operator S^ε, defined as $(S^\varepsilon \varphi)(x) = \int_Y \varphi(x - \varepsilon\omega)\, d\omega$, and the solutions N_γ, $|\gamma|=m$, and N_δ, $|\delta|=m+1$, to the above-mentioned cell problems. In [5], the following estimate is proved:

$$\|(\hat{A}_\varepsilon + 1)^{-1} + \varepsilon^m \mathcal{K}_m(\varepsilon) + \varepsilon^{m+1} \mathcal{K}_{m+1}(\varepsilon) - (A_\varepsilon + 1)^{-1}\|_{L^2(\mathbb{R}^d)\to H^m(\mathbb{R}^d)} \le C\varepsilon^2 \tag{8}$$

with the constant C depending only on the dimension d and constants λ_0 and λ_1 in (2). We have ε-uniform estimates (analogues of (6):

$$\|\varepsilon^m \mathcal{K}_m(\varepsilon)\|_{L^2(\mathbb{R}^d)\to H^m(\mathbb{R}^d)} \le c, \quad \|\varepsilon^m \mathcal{K}_{m+1}(\varepsilon)\|_{L^2(\mathbb{R}^d)\to H^m(\mathbb{R}^d)} \le c, \tag{9}$$

$$\|\mathcal{K}_m(\varepsilon)\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \le c, \quad \|\mathcal{K}_{m+1}(\varepsilon)\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \le c. \tag{10}$$

The estimates (10) show that the correcting terms in (8) cannot be transferred to the remainder, although they contain powers ε^m and ε^{m+1} with $m \ge 2$. On the other hand, simplifying (8) in view of (10), we deduce that, in the operator L^2-norm, $(A_\varepsilon + 1)^{-1} = (\hat{A}_\varepsilon + 1)^{-1} + O(\varepsilon^2)$, which extends (7) to the case of non-selfadjoint operators.

To prove our results, we use the shift method suggested by V. V. Zhikov in 2005.

REFERENCES

1. Pastukhova S. E. *Estimates in homogenization of higher-order elliptic operators.* Applicable Analysis. 2016. V. 95. P. 1449–1466.
2. Pastukhova S. E. *Operator error estimates for homogenization of fourth order elliptic equations.* St. Petersburg Math. J. 2017. V. 28. № 2. P. 273“289.
3. Pastukhova S. E. *L^2-approximation of resolvent in homogenization of higher order elliptic operators.* Journal of Mathematical Sciences. 2020. V. 251. № 6. P. 902“925.
4. Pastukhova S. E. *Resolvent L^2-approximation in homogenization of fourth order elliptic operators.* Sbornik: Mathematics. 2021 V. 212. № 1. P. 111“134.
5. Pastukhova S. E. *Improved approximations of resolvent in homogenization of higher order operators.* Journal of Mathematical Sciences. 2021. V. 259. № 2. P. 230“243.

SIMULATION OF MICROFRACTURES IN BLOCKY ROCK MASSIFS WITH THIN PLIABLE INTERLAYERS

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We analyze wave processes in blocky media applying different mathematical models, wherein elastic blocks interact with each other via compliant interlayers with complex rheological properties, taking into account cracking of a blocky structure along the interlayers. Material of interlayers before destruction can be elastic, viscoelastic, elastic-plastic or porous.

Models of blocky media are used to simulate wave processes in geomechanics and geodynamics. Such models are developed starting from the fundamental work [1], in which, based on the analysis of an experimental material, the natural lumpiness of rocks was established. Some basic results in this direction were obtained in [2]. As a result of modeling, a large system of nonlinear equations is obtained, which describes the dynamics of blocks with internal boundary conditions of their contact through interlayers. This system occurs to be practically inaccessible for research by analytical methods, and it requires the use of high performance computations.

Developing this approach, we construct different versions of the model [3, 4]: from the case of elastic, viscoelastic, or plastic interlayers to the case of a porous material in interlayers, where the pores collapse under application of compressive stresses is considered, and the case of a fluid-saturated porous material. In the present contribution, the model of a blocky medium is generalized to take into account the propagation of a system of cracks along interlayers. Internal boundary conditions on the crack edges are formulated as variational inequalities describing contact of blocks without friction.

As a criterion for destruction of interlayers at the initial stage we used various criteria by the level of instantaneous stresses and strains. However, such modeling can't be considered as adequate, since the process of dynamic destruction must include a preliminary stage of damage accumulation. Therefore, we apply in computations the Morozov–Petrov fracture criterion [5, 6], which is formulated in the integral form:

$$\frac{1}{\tau}\int\limits_{t-\tau}^{t}\frac{1}{d}\int\limits_{0}^{d}\sigma(x,\theta)\;dx\,d\theta \geqslant \sigma_c\,.$$

Here σ_c is the limit stress, τ is the microstructure parameter characterizing the time of damage accumulation, d is the block size. As an invariant of stress tensor, responsible for micro-failure, in the case of plane interlayers the following expression is taken:

$$\sigma = \sqrt{\frac{4}{9}\,\sigma_n^2 + \sigma_\tau^2}\,,$$

where σ_n and σ_τ are the normal and tangential stresses.

Using MPI (Message Passing Interface) technology, parallel software is developed for modeling the dynamics of blocky media with cracks in 2D formulation. Results of computations of the cracks growth caused by the rotation of blocks under the action of external pulse loads are presented.

Figure 1 demonstrates the crack propagation process in a blocky rock massif under intensive Π-shaped pulse loading at the central part of upper boundary. Computations were performed for the blocky massif consisting of 100 layers with 200 elastic-plastic blocks in each layer (10×10 cm is the size of each one) and pliable interlayers of thickness 1 mm.

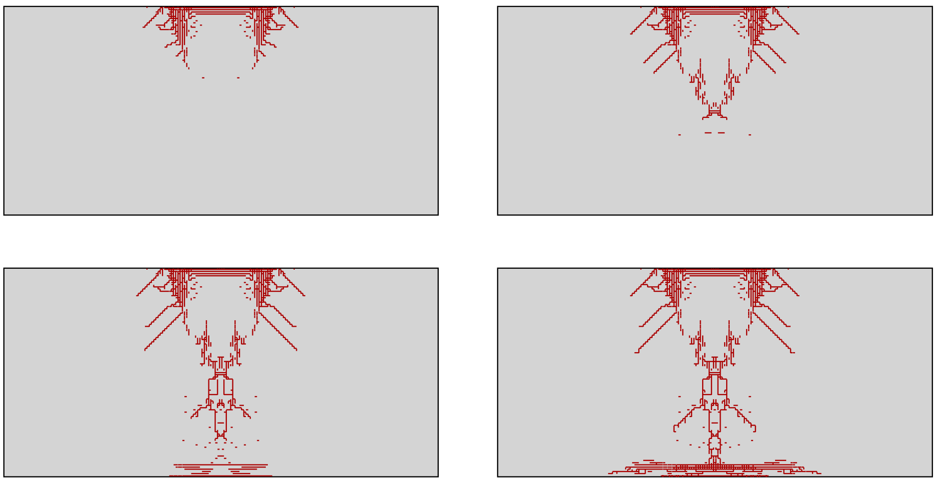

Figure 1: Configuration of fracture zones in the interlayers at different time moments

The clusters of the MVS series of the Institute of Computational Modeling SB RAS (Krasnoyarsk) and the Joint Supercomputer Center of the Russian Academy of Sciences (Moscow) were used for computations.

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REFERENCES

1. Sadovskii M. A. *Natural lumpiness of a rock.* Dokl. Akad. Nauk SSSR. 1979. V. 247. № 4. P. 829–831.
2. Aleksandrova N. I., Ayzenberg-Stepanenko M. V., Sher E. N. *Modeling the elastic wave propagation in a block medium under the impulse loading.* J. Min. Sci. 2009. V. 45. Art. 427.
3. Sadovskii V. M., Sadovskaya O. V. *Modeling of elastic waves in a blocky medium based on equations of the Cosserat continuum.* Wave Motion. 2015. V. 52. P. 138–150.
4. Sadovskii V. M., Sadovskaya O. V., Lukyanov A. A. *Modeling of wave processes in blocky media with porous and fluid-saturated interlayers.* J. Comp. Phys. 2017. V. 345. P. 834–855.
5. Petrov Yu. V., Morozov N. F. *On the modeling of fracture of brittle solids.* J. Appl. Mech. 1994. V. 61. Iss. 3. P. 710–712.
6. Morozov N., Petrov Yu. *Dynamics of Fracture.* Ser.: *Foundations of Engineering Mechanics.* Springer, Berlin – Heidelberg, 2000.

EVOLUTIONARY VARIATIONAL INEQUALITIES OF ELLIPTIC TYPE WITH IRREVERSIBILITY AND ENERGY BALANCE

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The Ambrosio-Tortorelli functional $AT_\varepsilon(u,z)$ is well known as a regularization of the Francfort-Marigo functional, which is introduced to describe crack initiation and propagation in brittle materials. Here u and z represent a displacement field and a phase field, respectively, in a brittle material $\Omega \subset \mathbb{R}^N$. A phase field $z = z(x,t)$ with values in $[0,1]$ is introduced to regularize the surface energy of crack sets, which is given as the lower-dimensional Hausdorff measure in the original Francfort-Marigo functional. We assume that crack occurs at points where z is close to 0, while there is no damage at points where z is close to 1. In [2], Giacomini constructed a flow of the displacement and phase fields in the frame for quasistatic evolution of cracks, as a continuous limit of a sequence of minimizers for AT_ε. Moreover, the flow satisfies the following three principles intrinsic to quasistatic evolution of cracks: irreversibility, quasistatic equilibrium and energy conservation law. On the other hand, several PDE models have also been studied; however, their solutions violate some of these principles due to some regularizations.

In this talk, we shall discuss the nonlinear PDE,

$$\partial I_{(-\infty,0]}\,(\partial_t z) - \Delta z + \sigma z \ni f \;\text{ in }\; \Omega \times (0,T), \tag{1}$$

where $T > 0$ is a constant, $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain, $f = f(x,t)$ and $\sigma = \sigma(x,t)$ are given functions and $\partial I_{(-\infty,0]}$ denotes the subdifferential operator of the indicator function $I_{(-\infty,0]}$ supported on $(-\infty, 0]$. Problem (1) is derived by a significant simplification of phase-field models for brittle fracture. However, its solution still fulfills the three principles mentioned above. Moreover, we note that (1) is equivalent to an evolutionary elliptic variational inequality.

Main results of this talk consist of two theorems: One is concerned with the well-posedness of problem (1) equipped with certain boundary and initial conditions under some assumptions on given data. The other one is concerned with the aforementioned three principles, which are originally used to characterize quasistatic evolution of cracks. In order to construct solutions to (1), we exploit a similar approximation of the equation to [1], where some parabolic equation with irreversibility is concerned; however, we shall employ a different strategy from [1] to derive energy estimates for (1), which is indeed an elliptic equation rather than parabolic one. Moreover, asymptotic behaviors of solutions as $t \to \infty$ will also be discussed, if time permits.

This talk is based on a joint work with Prof. Goro Akagi (Tohoku University). This work is partially supported by JSPS KAKENHI Grant Number 21J20732.

REFERENCES

1. Akagi, Goro, Kimura, Masato *Unidirectional evolution equations of diffusion type.* J. Differential Equations. 2019. V. 266. № 1. P. 1–43.
2. Giacomini, Alessandro *Ambrosio-Tortorelli approximation of quasistatic evolution of brittle fractures.* Calc. Var. Partial Differential Equations. 2005. V. 22. № 2. 129–172.

MULTIPHYSIC IMPERFECT INTERFACES MODELS: ASYMPTOTIC ANALYSIS AND NUMERICAL VALIDATION

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In the last decades, the interest in bonded structures, obtained by assembling different parts made of possibly different materials to compose a unique structure, is strongly increased. The advantage of such composites is that their mechanical performances and properties are designed to be superior to those of the constituent materials acting independently. In the present study, the attention is focused on a specific type of composite, constituted by two media, called the adherents, bonded together with a thin interphase layer, called the adhesive. The composite constituents are made of different linear multiphysic materials with highly contrasted constitutive properties. The study considers a generic multiphysic coupling in a very general framework and can be adapted to well-known multiphysic behaviors, such as piezoelectricity, thermo-elasticity, as well as to multifield microstructural theories, such as micropolar elasticity. The analysis has been carried out by means of asymptotic expansions method. By defining a small parameter ε, associated with the thickness and constitutive properties of the middle layer, an asymptotic analysis is performed. The middle layer thickness depends linearly on ε, while the multiphysic stiffness ratios between the adherents and the adhesive depend on ε^p. Three different limit models and their associated limit problems are characterized: the soft interface model, in which the constitutive coefficients depend linearly on ε; the hard interface model, in which the constitutive properties are independent of ε; the rigid interface model, in which they depend on $1/\varepsilon$. The asymptotic expansion method is reviewed by taking into account the effect of higher order terms and by defining a general multiphysic interface law which comprises the above aforementioned models. Finally, some numerical examples are given in order to show the efficiency of the proposed methodology in the case of piezoelectric and thermo-mechanical couplings. The numerical investigations are performed in the framework of the finite element method. In particular, finite elements are introduced to solve the fully 3D initial problem, considering a three-phase composite, and the limit 3D-2D problem with two layers with imperfect interface transmission conditions.

REGULARITY OF MINIMIZERS FOR PLATES WITH COHESIVE CRACKS

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We present some higher regularity results in Besov spaces for the minimizers in a variational model for Kirchhoff–Love elastic plates with vertical through cracks that accounts for a noninterpenetration constraint as well as cohesive forces acting between the crack faces.

OBSERVABILITY INEQUALITIES FOR ADVECTION EQUATIONS

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Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary $\partial\Omega$. We define the first-order hyperbolic operator $P := \partial_t + H(t) \cdot \nabla$ for a vector-valued function $H \in C^1([0,T];\mathbb{R}^d)$ and consider the Cauchy problem for a given $g \in L^2(\partial\Omega \times (0,T))$:

$$\begin{cases} Pu = \partial_t u + H(t) \cdot \nabla u = 0 & \text{in } \Omega \times (0,T), \\ u = g & \text{on } \partial\Omega \times (0,T). \end{cases}$$

We investigate a problem determining an initial value $u(\cdot,0)$ from the Cauchy data g in L^2 frameworks. I will show some results regarding uniqueness and stability for both cases when H is non-degenerate and degenerate. If time permits, I will mention observability inequalities for first-order symmetric hyperbolic systems as well.

These are joint works with Professor Masahiro Yamamoto (The University of Tokyo) and Professor Giuseppe Floridia (Mediterranea University of Reggio Calabria).

REFERENCES

1. P. Cannarsa, G. Floridia, and M. Yamamoto. Observability inequalities for transport equations through Carleman estimates. *Springer INdAM Ser.*, **32** 69–87, 2019.

2. G. Floridia and H. Takase, Observability inequalities for degenerate transport equations. to appear in *J. Evol. Equ.*
3. G. Floridia, H. Takase, and M. Yamamoto. A Carleman estimate and an energy method for a first-order symmetric hyperbolic system. *arXiv*:2110.12465.

ON THE JERKY CRACK GROWTH IN ELASTO-PLASTIC MATERIALS

Rodica Toader

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The purpose of this talk is to show that in some elasto-plastic materials cracks can grow only in an intermittent way. We consider a model for the quasistatic crack growth in pressure-sensitive elasto-plastic materials in the planar case and study the properties of the length $\ell(t)$ of the crack as a function of time. Under suitable technical assumptions on the crack path, we show that the monotone function ℓ is a pure jump function. This result was obtained in collaboration with G. Dal Maso, SISSA, Trieste, Italy.

REFERENCES

1. Dal Maso G., Toader R. *On the jerky crack growth in elastoplastic materials* Calc. Var. Partial Differential Equations (2020) 59:107 https://doi.org/10.1007/s00526-020-01752-2.
2. Dal Maso G., Toader R. *On the pure jump nature of crack growth for a class of pressure-sensitive elasto-plastic materials* Nonlinear Anal. 214 (2022), Paper No. 112539, https://doi.org/10.1016/j.na.2021.112539.

ON DETERMINATION OF COEFFICIENT OF ELLIPTIC EQUATION VIA SINGLE PARTIAL BOUNDARY MEASUREMENT

Igor Trushin

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We investigate an inverse problem to identify coefficient of elliptic equation via single Dirichlet-Neumann partial boundary measurement in the case of rectangular (cubic) domain and conductivity depending on only one variable. Joint work with H.Kang (Inha University, Korea) and J-Y.Lee (Ewha Womans University, Korea)

METHODS OF THE KANTOROVICH-VLASOV TYPE FOR THE ANALYSIS OF POROUS FUNCTIONAL-GRADIENT NANOPLATES TAKING INTO ACCOUNT PHYSICAL NONLINEARITY SUBJECTED TO TEMPERATURE FIELD

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Mathematical model of Kirchhoff nanoplates made of a porous functionally graded material (PFGM) subjected to temperature are derived. For modeling size-dependent factors of the composite nanoplate, modified couple stress theory [1] was use. Lagrange principle is used for obtaining the governing equations of the composite nanoplate. The temperature field is determined from the solution of the three-dimensional heat conduction equation by the finite element method. We consider the special case when the Young's modulus $E = E(z, e_i)$ and the Poisson ratio $\nu = \nu(z, e_i)$ are functions of the thickness variable z and the intensity of deformation e_i. The distributions of porosity are given respectively by three different types of porosity [2], in which the porosity and FG of the material plate are defined using the power functions: (i) uniform porosity (U-PFGM), (ii) reduced porosity from the top and bottom surfaces to the center (X-PFGM), and (iii) increased porosity at the top and bottom (the surfaces shown in Fig. 2 are viewed down to the center (O-PFGM)). The Poisson ratio and Young's modulus of a porous functionally graded material (PFGM) of a nanoplate associated with different porosity distributions can be described in the following ways:

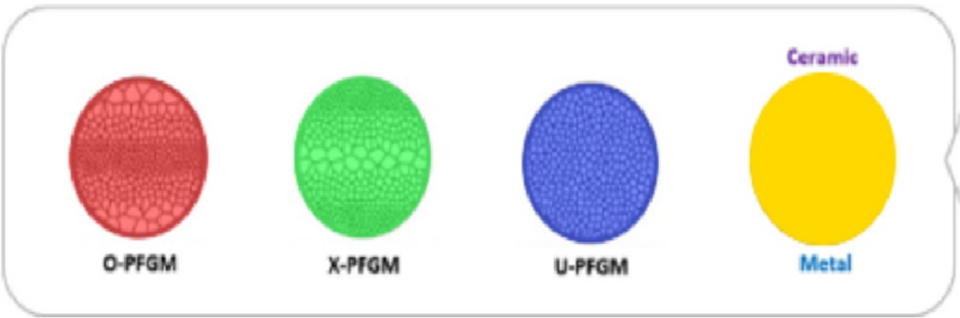

Figure 2: Schemes of porous material

i) For U-PFGM pattern:

$$\begin{aligned} E(z,e_i) &= (E_c - E_m(z,e_i))(1/2 + z/h)^k + E_m(z,e_i) - (E_c + E_m(z,e_i))\Gamma^*/2, \\ \nu(z,e_i) &= (\nu_c - \nu_m(z,e_i))(1/2 + z/h)^k + \nu_m(z,e_i) - (\nu_c + \nu_m(z,e_i))\Gamma^*/2, \\ \alpha_T(z,e_i) &= (\alpha_{Tc} - \alpha_{Tm}(z,e_i))(1/2 + z/h)^k + \alpha_{Tm}(z,e_i) - (\alpha_{Tc} + \alpha_{Tm}(z,e_i))\Gamma^*/2, \end{aligned} \quad (1)$$

ii) X-PFGM pattern:

$$\begin{aligned} E(z,e_i) &= (E_c - E_m(z,e_i))(1/2 + z/h)^k + E_m(z,e_i) - (E_c + E_m(z,e_i))(1/2 - |z|/h)\Gamma^*, \\ \nu(z,e_i) &= (\nu_c - \nu_m(z,e_i))(1/2 + z/h)^k + \nu_m(z,e_i) - (\nu_c + \nu_m(z,e_i))(1/2 - |z|/h)\Gamma^*, \\ \alpha_T(z,e_i) &= (\alpha_{Tc} - \alpha_{Tm}(z,e_i))(1/2 + z/h)^k + \alpha_{Tm}(z,e_i) - (\alpha_{Tc} + \alpha_{Tm}(z,e_i))(1/2 - |z|/h)\Gamma^*, \end{aligned} \quad (2)$$

iii) O-PFGM pattern:

$$\begin{aligned} E(z,e_i) &= (E_c - E_m(z,e_i))(1/2 + z/h)^k + E_m(z,e_i) - (E_c + E_m(z,e_i))|z|\Gamma^*/h, \\ \nu(z,e_i) &= (\nu_c - \nu_m(z,e_i))(1/2 + z/h)^k + \nu_m(z,e_i) - (\nu_c + \nu_m(z,e_i))|z|\Gamma^*/h, \\ \alpha_T(z,e_i) &= (\alpha_{Tc} - \alpha_{Tm}(z,e_i))(1/2 + z/h)^k + \alpha_{Tm}(z,e_i) - (\alpha_{Tc} + \alpha_{Tm}(z,e_i))|z|\Gamma^*/h, \end{aligned} \quad (3)$$

In the above, Γ^* represents indicator of porosity [2], ν_c, ν_m- the Poisson ratio, E_c, E_m - Young's modulus and α_{Tc}, α_{Tm} stands for the thermal expansion coefficients associated with the ceramic and metal phases functionally graded material (FGM), k represents the gradient index of material property. It shows the ratio of the volumetric fractions of the material (in particular, ceramics at the top and metal at the bottom). If $k = 0$, then no pores. The power coefficient k takes the values $0.2 \leq k \leq 5$. where as $E_c = 210GPa$, $E_m = 70GPa$, $\nu_c = 0.24$, $\nu_m = 0.35$, $\alpha_{Tc} = 23 \cdot 10^{-6} 1^0 C$, $\alpha_{Tm} = 24 \cdot 10^{-6} 1^0 C$, and we fixed $\Gamma^* = 0.4$, $k = 1$ while carrying the numerical simulation.

The presented experimental data refer to the case of an elastic material. In the above formulas (1 - 3), a new assumption was presented. Namely, the metal has the property of physical nonlinearity. According to the deformation theory of plasticity, information is required about dependencies stress intensity from deformations $\sigma_i(e_i)$. For porous nanoplates, Kantorovich - Vlasov Methods (KVM), Variational Iteration Method (VIM), Vaindiner Method (VaM) and Agranovskii-Baglai-Smirnov (ABSM) are applied [3]. At each step of loading, the iterative procedure of the method of variable parameters of elasticity Birger [4] was constructed. Thus, at each loading step, we got 3 iterations, nested one into the other. This gives possibility get reliable solution for porous functional-gradient nanoplates, which most closely describe the real working conditions.

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REFERENCES

1. Yang F. A. C. M., et al. *Couple stress based strain gradient theory for elasticity.* International journal of solids and structures. 2002. V. 39. № 10. P. 2731-2743.
2. Fan Fan , et al. *Modified couple stress-based geometrically nonlinear oscillations of porous functionally graded microplates using NURBS-based isogeometric approach.* Computer Methods in Applied Mechanics and Engineering. 2020. V. 372. № 113400.
3. Awrejcewicz, J., Krysko-Jr. V. A., Kalutsky L. A., Zhigalov M .V., Krysko V. A. *Review of the Methods of Transition from Partial to Ordinary Differential Equations: From Macro- to Nano-structural Dynamics.* Arch Computat Methods Eng. 2021. V. 28. P. 4781-4813.
4. Birger I. A. *Some general solution methods for problems in plasticity theory.* Appl Mat Mech. 1951. V. 15. № 6. P. 765-770. (in Russian)